Frequency Response

1 Nyquist (Polar) Plot

- Polar plot is a plot of magnitude of $G(j\omega)$ versus the phase of $G(j\omega)$ in polar coordinates as shown in Figure 1.

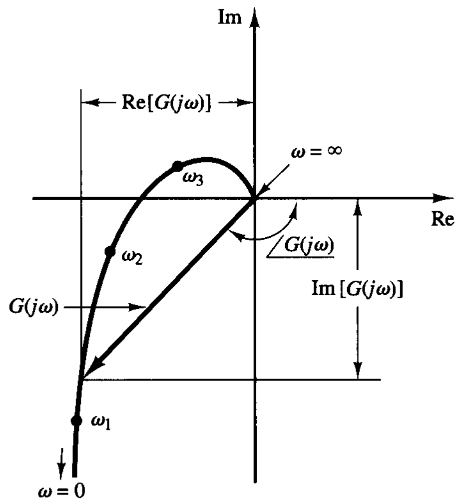


Figure 1: Polar Plot

- In polar plot, the positive angle is measured counter-clockwise direction.
- The magnitude is determined in standard scale (not Decibel scale)
- **ADV:** capture the system behavior over the entire frequency range in a single plot
- Disadv: hides the impact of individual components of the open-loop transfer function.

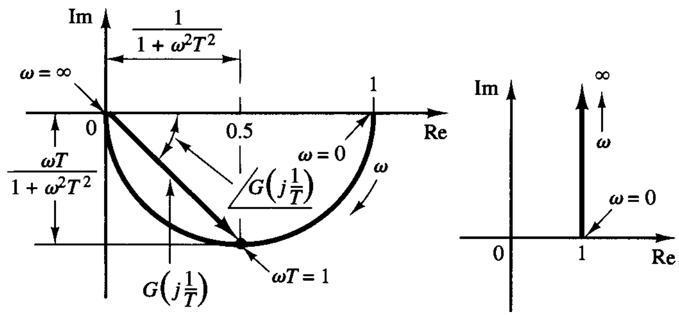
1.1 Transfer Function Component Representation

1.1.1 Integral and Derivative Factors $(j\omega)^{\pm 1}$

- $G(j\omega) = 1/j\omega \Rightarrow \begin{array}{c} mag\, 1/\omega \\ \phi = -90 \end{array}$. The locus is the negative frequency axis
- $G(j\omega) = 1/j\omega \Rightarrow \begin{array}{c} mag\, \omega \\ \phi = -90 \end{array}$. The locus is the positive frequency axis

1.1.2 First Order Factors $(1 + j\omega T)^{\pm 1}$

- $G(j\omega) = 1/(1 + j\omega T) \Rightarrow \begin{array}{c} mag = 1/\sqrt{1 + (\omega T)^2} \\ \phi = -tan^{-1}\omega T \end{array}$.

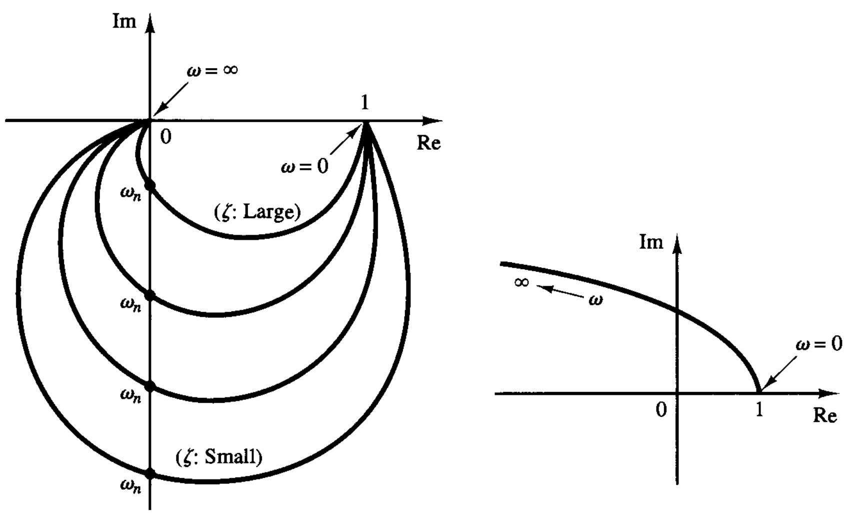


(a) $1/(1 + j\omega T)$ (b) (1+jωT)

Figure 2: Polar Plot

1.1.3 Second Order Factors

- $|G| = \sqrt{(1 - \left(\frac{\omega}{\omega_n}\right)^2)^2 + (2\zeta\frac{\omega}{\omega_n})^2}$, $PH(G) = -tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{w}{w_n}\right)^2}$



(a) $1/(1 + 2\zeta\frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2)$ (b) $(1 + 2\zeta\frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2)$

Figure 3: Polar Plot of Quadratic Systems

- The plot significantly depends on ζ as shown in figure
- The phase is exactly -90 at ω_n
- As the damping ration increases, the roots become real and the impact of the larger root become negligible. In this case, the system behaves like a first-order system.

Example

Plot the Nyquist plot of $G(s) = \frac{1}{s(1+Ts)}$
The presence of the integral term has an impact the should be considered
$G(j\omega) = \frac{-T}{1+\omega^2T^2} - j\frac{1}{\omega(1+\omega^2T^2)}$

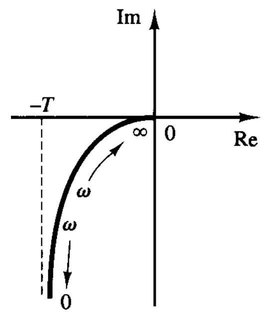


Figure 4: Nyquist plot of $G(s) = \frac{1}{s(1+Ts)}$

1.1.4 Transport Lag

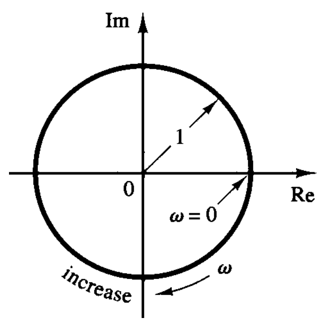


Figure 5: Transport Lag

Example: Plot Nyquist plot of $G(s) = \frac{e^{-sT}}{1+sT}$

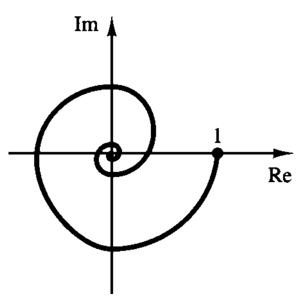
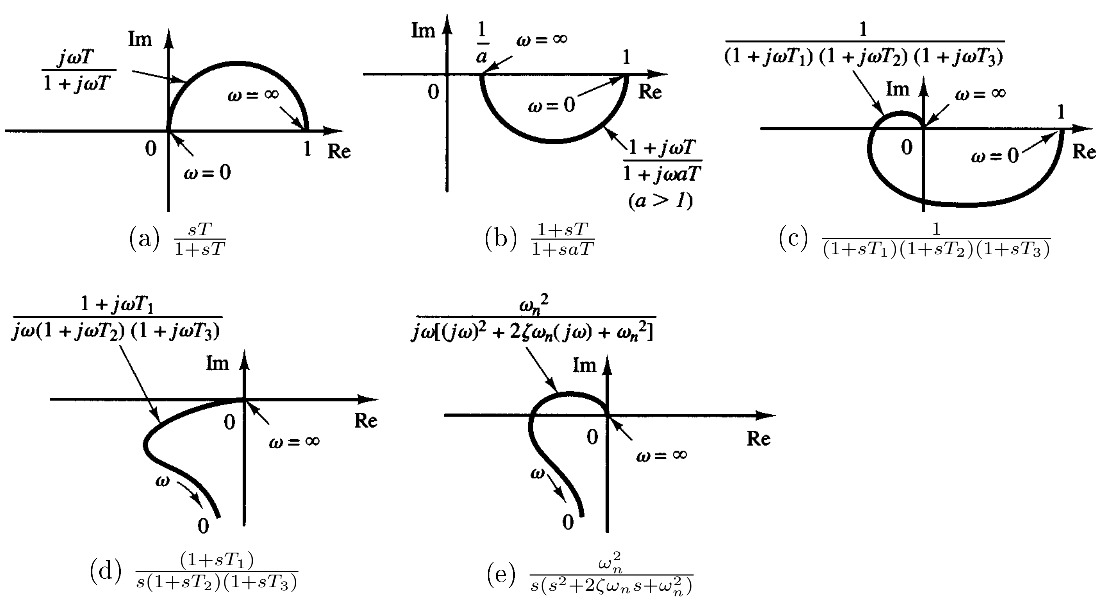


Figure 6: Nyquist plot of $G(s) = \frac{e^{-sT}}{1+sT}$

1.2 Notes on general polar plots

- For physical realizable systems, the order of the denominator is larger than or equal to that of the numerator of the transfer function.

- Type 0 systems:
  - finite starting point on the positive real axis
  - The terminal point is the origin tangent to one of the axis
- Type 1 Systems
  - starting at infinity asymptotically parallel to -ve imaginary axis
  - Also the curve converges to zero tangent to one of the axis
- Type 2 systems
  - the starting magnitude is infinity and asymptotic to -180
  - Also the curve converges to zero tangent to one of the axis
- Examples



(a) $\frac{sT}{1+sT}$ (b) $\frac{1+sT}{1+saT}$ (c) $\frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$

(d) $\frac{(1+sT_1)}{s(1+sT_2)(1+sT_3)}$ (e) $\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$

Figure 7: Nyquist plot Examples

1.3 Procedure of Nyquist Plot

1. express the magnitude and phase equations in terms of ω
2. Estimate the magnitude and phase for different values of ω.
3. Plot the curve and determine required performance metrics

Example: Draw Nyquist plot for $G(s) = \frac{20(s^2+s+0.5)}{s(s+1)(s+10)}$

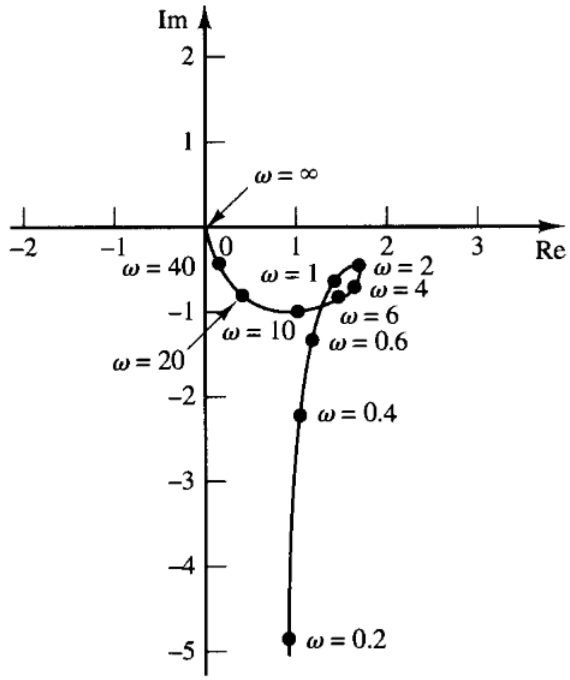
Solution

$|G(j\omega)| = \frac{20\sqrt{(0.5-\omega^2)^2+\omega^2}}{\omega\sqrt{(1+\omega^2)(100+\omega^2)}}$

$$\angle G(j\omega) = tan^{-1}\frac{\omega}{0.5-\omega^2} - 90 - tan^{-1}\omega - tan^{-1}(\omega/10)$$

ω	$\lvert G(j\omega)\rvert$	$\angle G(j\omega)$
0.1	9.952	-84.5
0.2	4.91	-78.9
0.4	2.4	-64.5
0.6	1.7	-47.53
1	1.573	-24.15
2	1.768	-14.5
6	1.8	-22.25
10	1.407	-45.03
20	0.893	-63.44
40	0.485	-75.96

(a) Estimated values



(b) Polar Plot

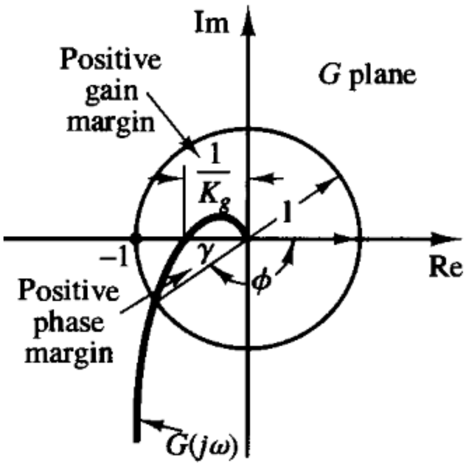
Table 1: Nyquist Plot for $G(s) = \frac{20(s^2+s+0.5)}{s(s+1)(s+10)}$

1.4 Nyquist plot using Matlab

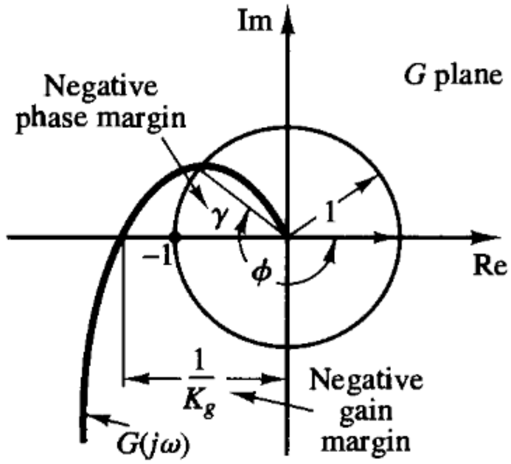
```
num=[0 0 25];
den=[1 4 25];
nyquist(num,den);
```

1.5 Relative Stability Analysis using Nyquist Plot

- On investigating stability, one should be more have an accurate around $|G(j\omega)| = 1$ and $\angle G(j\omega) = -180$ to obtain more accurate results for gain margin and phase margin.



(a) Stable System



(b) Unstable System

Figure 8: Relative Stability Analysis using Nyquist Plot

2 Nyquist Stability Criteria

- It is a graphical technique for determining the stability of linear time-invariant system
- Considering the closed loop system shown in Figure 9,

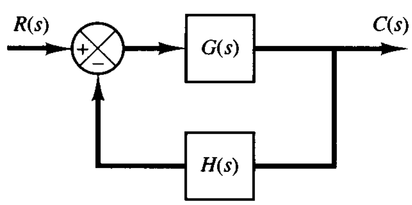


Figure 9: Closed Loop system

the transfer function is expressed as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

For stability, all the roots of the characteristic equation $1+GH(s) = 0$ must lie in the left-half plane.

- Note that open loop transfer function of a stable system may have poles in the left half plane
- Nyquist stability criteria relates the open-loop transfer functions and the poles of the characteristic function.

2.1 Cauchy's argument principle

- Let $F(s)$ denotes a complex function that is a ration of two polynomials, i.e. $F(s) = \frac{polynomial}{polynomial}$

- Let (x, y) represent a point in the s-plane.
- By direct substitution of (x, y) in $F(s)$, $F(s)$ will take a complex value
- Consider now a contour Γs drawn in the complex s-plane, the by substituting of all points on the contour in $F(s)$, we get another contour in $F(s)$ plane.
- The process described above is called contour mapping
- Cauchy argument principal states that
  - a contour encompassing BUT NOT PASSING through any number of zeros and poles of a function F(s), can be mapped to another plane (the F(s) plane) by the function F(s).
  - The resulting contour ΓF(s) will encircle the origin of the F(s) plane N times, where N = Z − P, where Z and P are respectively the number of zeros and poles of F(s) inside the contour Γs.
- Note that we count encirclements in the F(s) plane in the same sense as the contour Γs and that encirclements in the opposite direction are negative encirclements.

2.2 Nyquist Stability Criteria

- Nyquist stability criteria is based on **Cauchy's argument principle** of complex variables.
- For stability analysis of ***closed loops systems***, the chosen complex contour should cover the entire right half plane.
- Such path is called ***Nyquist path*** and consists of a semicircle starting at $-\infty$ to ∞.
- By Cauchy Argument Principle, the mapped Nyquist contour in $GH(s)-$plane makes a number of clock-wise encirclements around the origin equals the number of zeros of GH(s) in the right-half complex plane minus the poles of GH(s) in the right-half complex plane.
- For stability analysis, we need to check if $1+GH(s)$ has any zeros in the RHP or not. Noting that only difference between mapping $1+GH(s)$ and mapping $GH(s)$ is the addition of one, which is equivalent to a linear shift in the origin.
- Hence, we can use the mapped Nyquist contour of the open loop to to investigate the stability of the closed loop system.
- Before delving into the details of the stability analysis procedure, it is important to point out the following facts
  - the zeros of $1 + GH(s)$ are the poles of the closed-loop system, and
  - the poles of $1 + GH(s)$ are same as the poles of $G(s)$

2.2.1 Application of Nyquist Stability Criteria

- let P be the number of poles of GH(s) [same as 1+GH(s)] encircled by Γs (in other words poles in the RHP for Nyquist path), and
- Z be the number of zeros of 1 + GH(s) encircled by Γs. Z is the number of poles of the closed loop system in the right half plane.
- The resultant contour in the GH(s)-plane, ΓGH(s) shall encircle (clock-wise) the point (− 1 + j0) N times such that **N = Z − P**.
- Stability Test
  - **Unstable open-loop systems (P>0)**, we must have Z=0 to ensure stability. Hence, we should have N=-P, i.e. ***counter clockwise encirclements***. If $N \neq -P$, then some of the unstable poles have not moved to the LHP.
  - **Stable open-loop systems (P=0)**, therefore N=Z. For stability, there must be ***no encirclemen***t to -1. In this case, it is sufficient to consider only the positive frequency values of ω.
- If Nyquist plot passes through -1+j0 point, this indicates that the system has close loop poles on $j\omega$ axis

Example 4

Investigate the stability of $GH(s) = \frac{K}{(1+T_1 s)(1+T_2 s)}$ using Nyquist Stability Criteria
Solution

- The Nyquist plot of the open loop transfer function is shown in Figure 10

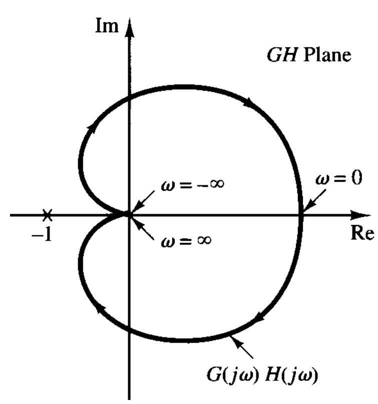


Figure 10: Nyquist plot of $GH(s) = \frac{K}{(1+T_1 s)(1+T_2 s)}$

- $P = 0 \Longrightarrow$we need N=Z=0 for stability
- As shown in the figure, there is no encirclement for -1. Hence, the closed loop system is stable for any positive value of K, T_1, T_2

2.2.2 Nyquist stability for GH(s) with poles and or zeros on $j\omega$ axis

- Typically, the Nyquist path should not go through any pole or zero. Hence, the Nyquist path should be slightly modified to avoid this situation.
- Nyquist path is altered by allowing a semi-circle detour with an infinitesimal radius around the origin.
- The small semi-circle is represented using magnitude and phase $\epsilon e^{j\vartheta}$.
  - Note that for type-1 systems,$\lim_{s\to\epsilon e^{j\vartheta}} GH(s) = \frac{1}{\epsilon}e^{-j\vartheta}$
  - Note that for type-1 systems,$\lim_{s\to\epsilon e^{j\vartheta}} GH(s) = \frac{1}{\epsilon^2}e^{-j2\vartheta}$
- Example $G(s) = K/[s(1+Ts)]$

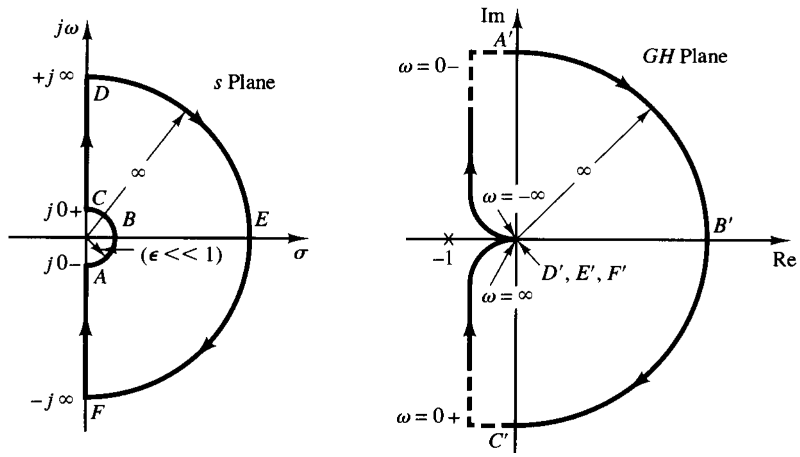


Figure 11: Modified Nyquist Path GH(s)=$K/[s(1+Ts)$

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  - P=0,
  - No encirclements form contour mapping N=0,
  - Z=P+N=0 ⇒the system is stable.
- Example: $G(s) = K/[s^2(1+Ts)]$

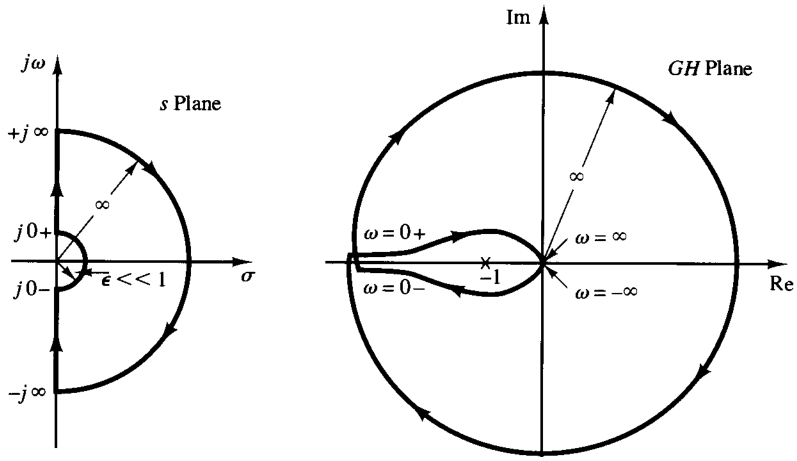

Figure 12: Modified Nyquist Path $G(s) = K/[s^2(1 + Ts)]$

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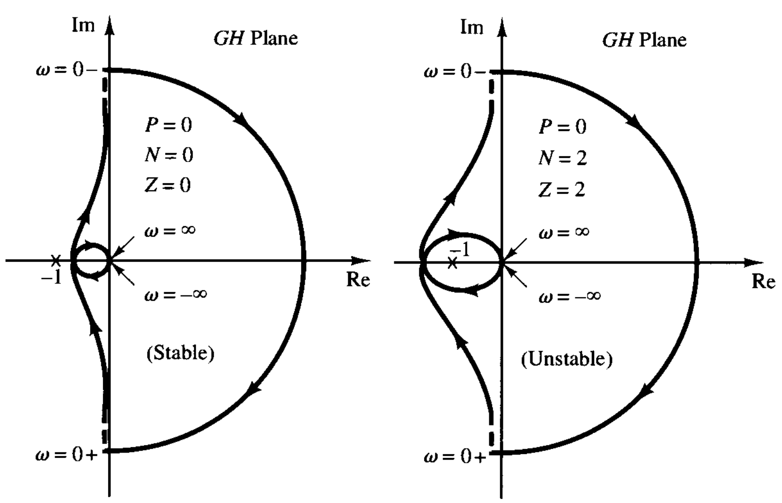
- P=0 for positive T
- Two clockwise encirclements N=2,
- ∴Z=N+P=2 ⇒there exist two zeros for the characteristic equations in the RHP.
- Hence, the system is unstable.

Example 5

Investigate the stability of $GH(s) = \frac{K}{s(1+T_1 s)(1+T_2 s)}$ using Nyquist Stability Criteria

Solution

- The Nyquist plot of the open loop transfer function is shown in Figure 13



(a) Small K (b) Large K

Figure 13: Nyquist plot of $GH(s) = \frac{K}{(1+T_1 s)(1+T_2 s)}$

- $P = 0 \Longrightarrow$we need N=Z=0 for stability

Example 6

Investigate the stability of $GH(s) = \frac{K(1+T_2 s)}{s^2(1+T_1 s)}$ using Nyquist Stability Criteria for positive values of T_1 and T_2.

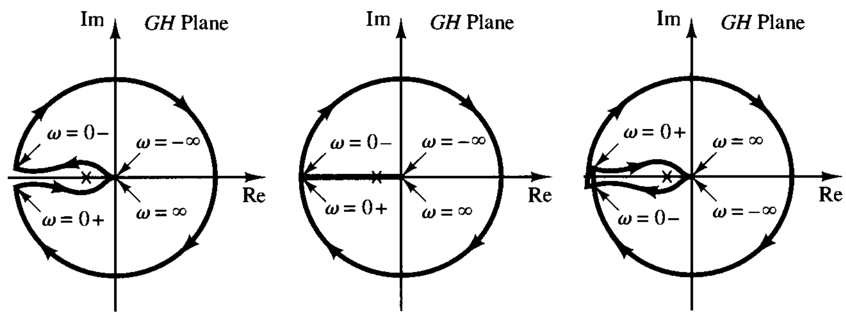


Figure 14: $GH(s) = \frac{K(1+T_2 s)}{s^2(1+T_1 s)}$ Stability Analysis

2.2.3 Conditionally Stable Systems

- Figure 15 shows an example of a system that may encircle -1 depending on the value of the system gain (or input signal amplitude). For the shown system, the increase or decrease of the system gain would lead to an unstable behavior.

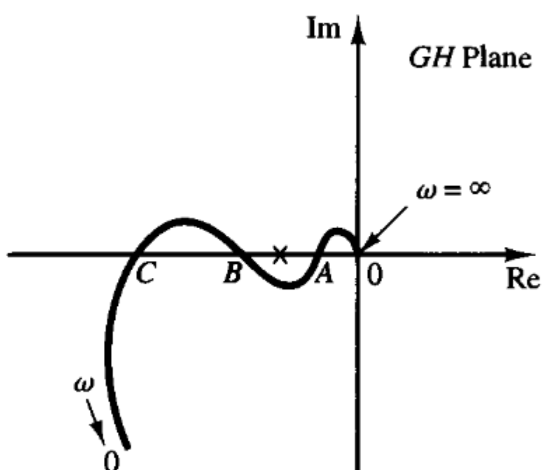


Figure 15: Conditionally Stable systems

- **Conditionally stable systems** are systems that are stable for a specific range of system gain or input signal amplitude.
- Note that large signal amplitude may also drive the system to the saturation region due to inherent system non-linearities.

2.2.4 Multiple Loop systems

- Consider the system shown in Figure 16. The inner loop transfer function $G(s) = \frac{G_2(s)}{1+G_2(s)H_2(s)}$.

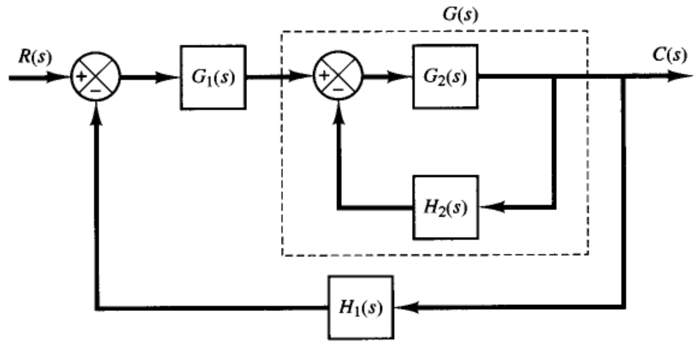


Figure 16: Multiple Loop Systems

- To analyze this system, we can apply Nyquist stability criteria recursively.
  - First, we apply the criteria on the inner loop to identify the zeros of $1 + G_2(s)H_2(s)$. These zeros are poles in the overall system open-loop transfer function $G_1(s)G(s)H_1(s)$.
  - Second, we perform the stability analysis for the overall system open-loop transfer function $G_1(s)G(s)H_1(s)$.

Example 7

For the system shown in Figure 17, determine the range of K for a stable system.

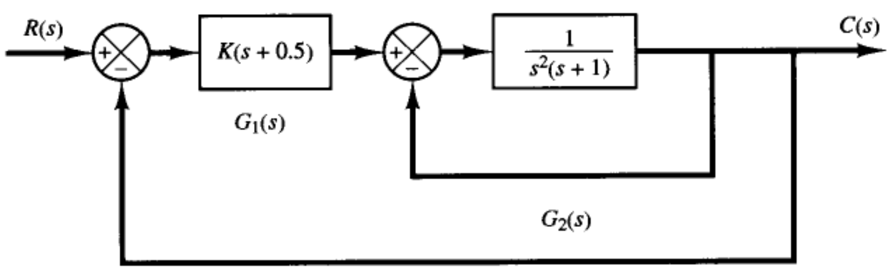


Figure 17: Example 7

Solution

- First, we determine the number of zeros of $(1 + G_2(s))$in the RHP. Note that P=0, From Example 6, $T_2 < T_1$, we have an unstable system with N=2. Hence, Z=N-P=2-0=2. The inner loop has two poles in the RHP.
- Note that one can get the same conclusion about the inner loop using another tool such as Routh stability criteria.
- Proceeding to the full system, one can plot Nyquist diagram for K=1 as shown in Figure

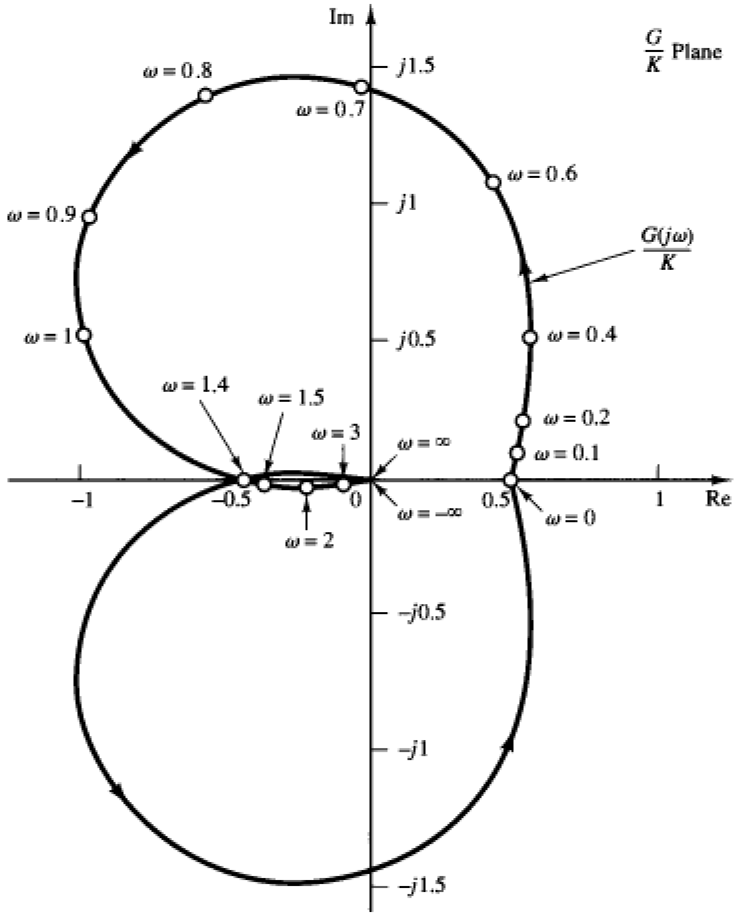


Figure 18: Nyquist plot for Example 7

- Initially, we have P=2 from the inner loop.
- For stability, we need N=2 such that Z=N-P=0.
- Hence, we need ***K>2.***

3 Final notes about frequency response

3.1 Relation between frequency response and time-response

Generally, it is easier to design a system using frequency domain tools. However, it is typical in many applications that the transient response to aperiodic signals rather than the steady state response of a sinusoidal input is of interest. Hence, there is always in studying the relation between the frequency response and the transient response.

3.1.1 Relation in second order systems

- The closed loop of the second order system shown in Figure 19 is $\frac{c(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n+\omega_n^2}$.

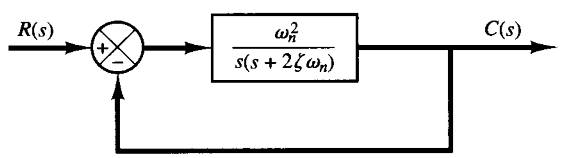


Figure 19: Second order system

- This system has the complex poles $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ for $0 < \zeta < 1$

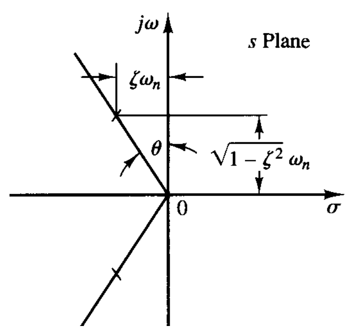


Figure 20: Complex poles

- Additionally, the closed loop frequency response is

$$\frac{C(j\omega)}{R(j\omega)} = \frac{1}{\left[1-\frac{\omega^2}{\omega_n^2}\right]+j2\zeta\frac{\omega}{\omega_n}} = Me^{j\alpha}$$

- for $0 < \zeta < 0.707$, the maximum value of M, denoted as M_r, occurs at the resonance frequency $\omega_r = \omega_n\sqrt{1-2\zeta^2} = \omega_n\sqrt{cos2\vartheta}$. At this frequency the maximum magnitude [resonance peak magnitude] is

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{1}{sin2\vartheta}$$

- The magnitude of the resonance peak is an indication for the system relative stability. A large resonance peak indicates the existence of a dominant pair of complex poles with a small damping ratio. Such poles may lead to an undesirable transient response.

- Note that the max resonance peak and the resonance frequency can be easily measured during system experimentation.
- Considering the open loop transfer function of the system shown in Fig 19, one can determine ω_{gc} as

$$\omega_{gc} = \omega_n \sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

  and consequently the phase margin can be calculates as

$$\begin{aligned} PM &= 180 + \angle G(j\omega_{gc}) \\ &= tan^{-1} \frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}} \end{aligned}$$

  Note that the PM is only function of the damping ratio and can be plotted as shown in Figure

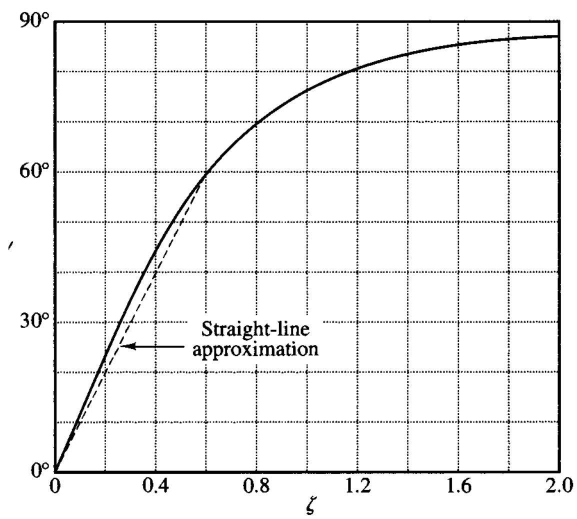


Figure 21: Phase Margin vs. damping ratio in second order systems

- The unit step response of second order system shown in Fig 19 can be characterized using different parameters
  - The damping frequency $\omega_d = \omega_n \sqrt{1-\zeta^2} = \omega_n cos\vartheta$
  - the maximum overshot $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$
- To sum up the main results
  - PM and damping ratio are linearly proportional for small damping ratios and their relation can be approximated as

$$\zeta = \frac{PM(deg)}{100}$$

    note that this equation is applied as a rule of thumb for any system with a dominant second order pole to anticipate the transient response from our knowledge of frequency response.

- The values of ω_r and ω_d are approximately equal for small values of the damping ration ζ. Hence, ω_r can be considered an indication for the speed of damping of the transient response.
- There is also a correlation between M_p and M_r as shown in figure

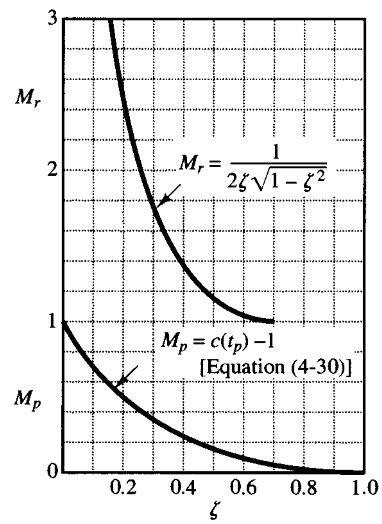


Figure 22: Relation between M_p and M_r

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3.1.2 General System

- In general systems, obtaining the time-frequency response relationship is not as direct as it is in second order systems
- Typically, the addition of any poles may change the correlation between step transient response and frequency response
- However, the derived results for second order systems may be applicable to higher order systems in the presence of a dominant second order system poles
- For an LTI higher order systems with a dominant second order pole, the following relationships generally exists
  - The value of M_r is indicative for the relative stability. A satisfactory performance is attained for $1 < M_r < 1.4$, which corresponds to a damping ration of $0,4 < \zeta < 0.7$. A large M_r indicates a high overshot and slow damping.
  - If the system is subject to noise signals whose frequency are near to the resonance frequency ω_r, the noise will be amplified in the output causing a serious problem.
  - The magnitude of the resonance frequency ω_r indicates the speed of the transient response. Large ω_r indicates faster time response [smaller rise and settling times]
  - The resonant peak frequencyω_r and the damped natural frequency ω_d of unit step response are very close to each other for lightly damped systems.