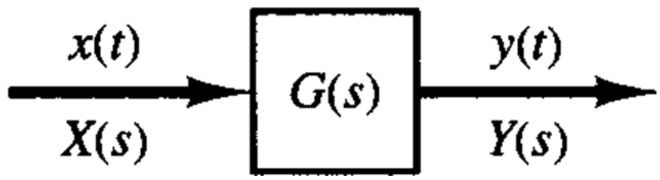
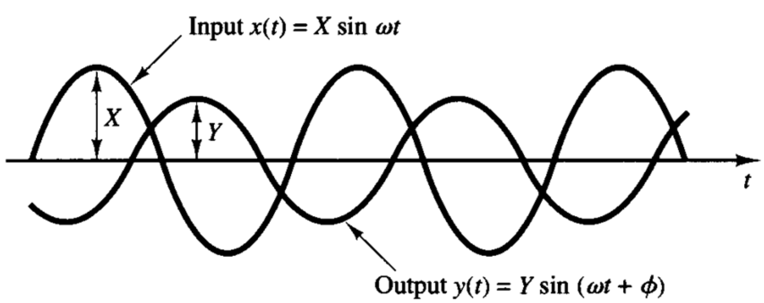
Frequency Response

1 Introduction

- **Frequency Response** is the response of a system to a sinusoidal input



- For a stable linear time-invariant system, the system produces a scaled shifted sinusoidal of a sinusoidal input



$$x(t) = XSin\omega t \qquad\qquad y(t) = Ysin(\omega t + \phi)$$

1.1 Notation and Terminology

$$x(t) = XSin\omega t \qquad\qquad y(t) = Ysin(\omega t + \phi)$$

$G(s) = \frac{Y(s)}{X(s)} \Longrightarrow$ System Transfer Function

$Y = X\,|G(j\omega)| \Longrightarrow$Output magnitude

$\phi = tan^{-1}\frac{Img(G(j\omega)}{Re(G(j\omega)} \Longrightarrow$ Phase shift of the output sinusoid from the input sinusoid

$|G(j\omega)| = \frac{Y}{X} \Longrightarrow$ Amplitude Ratio of the output sinusoid to the input sinusoid

$PH[G(j\omega)] = \phi$

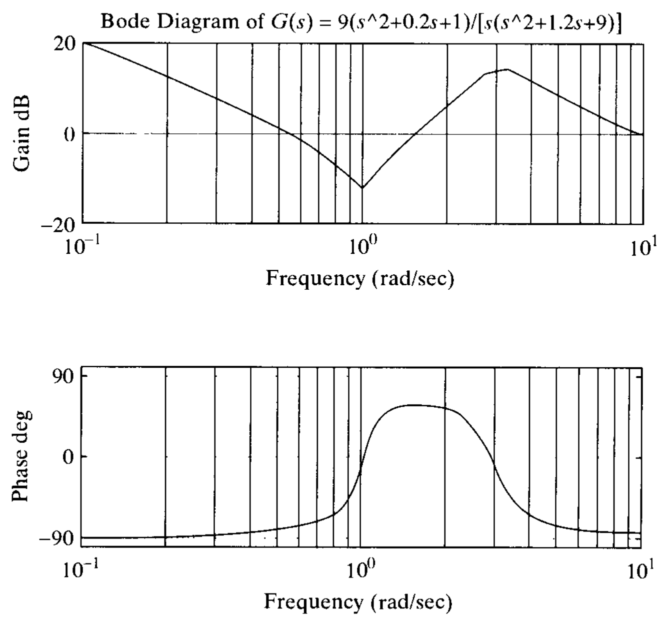
- Negative phase angle is known as ***Phase lag systems*** while positive phase angle corresponds to ***phase lead systems***

1.2 Graphical Representation of Frequency Response

- Three Common representation of sinusoidal transfer function
  - Bode plot
  - Nyquist plot
  - log-magnitude versus phase plot

2 Bode Plot

- Bode plot is a (semi-log) diagram consists of two plot
  - the transfer function magnitude vs. frequency
  - phase transfer function angle vs. frequency



- The gain magnitude is many times expressed in terms of **decibels (dB)**

$$dB = 20log_{10}A$$

  where A is the amplitude or gain

- A **decade** is defined as any 10-to-1 frequency range
- An **octave** is any 2-to-1 frequency range

$$20dB/decade = 6dB/octave$$

- **An important advantage of bode plot is that multiplication of magnitude can be converted into addition**.

- Another advantage is the ability to sketch using a simple method based on asymptotic approximation. The exact curve can be obtained using some known corrections.
- Bode diagrams can be easily used to obtain the transfer function from the experimentally obtained bode plots.

2.1 Basic factors of Bode Plots

a general transfer function that can be considered can be expressed as

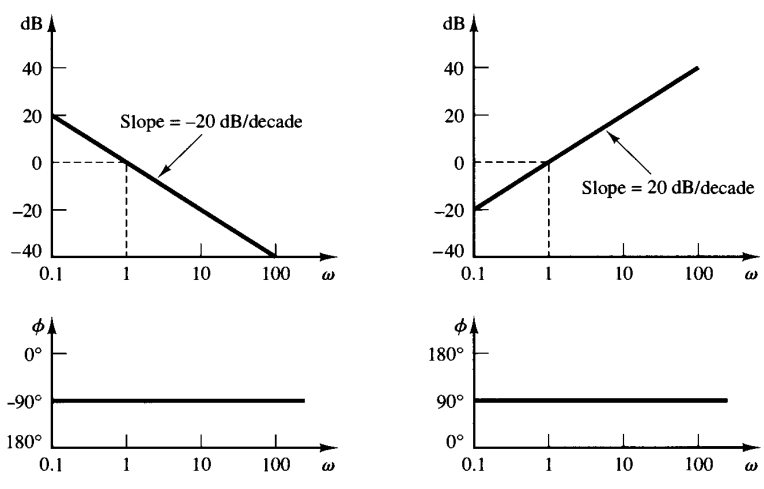
$$G(s) = \frac{K(1 + j\omega T_z)\left[1 + 2\zeta\left(j\omega/\omega_{nz}\right) + \left(\omega/\omega_{nz}\right)^2\right]}{(j\omega)^n(1 + j\omega T_p)\left[1 + 2\zeta\left(j\omega/\omega_{np}\right) + \left(j\omega/\omega_{np}\right)^2\right]}$$

2.1.1 Constant (K)

- $|G(j\omega)| = constant$
- $\phi = 0$
- The effect of the gain is just raising or lowering the bode magnitude curve

2.1.2 Integral and Derivative Terms $(jw)^{\pm 1}$

- $1/(j\omega) \Longrightarrow \begin{array}{c} Magnitude \rightarrow -20log\omega\ dB \\ Phase \rightarrow -90deg \end{array}$
- $(j\omega) \Longrightarrow \begin{array}{c} Magnitude \rightarrow 20log\omega\ dB \\ Phase \rightarrow 90deg \end{array}$
- Note that the magnitude equals ZERO at $\omega = 1$



Bode diagram of $G(j\omega) = 1/j\omega$

Bode diagram of $G(j\omega) = j\omega$

2.1.3 First Order Factors $(1 + jwT)^{\pm 1}$

- $(1 + jwT) \Longrightarrow \begin{array}{c} |G| = 20log(\sqrt{1 + (\omega T)^2}) \\ PH(G) = tan^{-1}(\omega T) \end{array}$
- $\frac{1}{T}$ is called the corner frequency divides the curve into two regions ($\phi = tan^{-1}1 = 45deg$)
- $\omega \lll 1/T$: magnitude equals ZERO
- $\omega \ggg 1/T$: magnitude drops 20dB/decade
- CORRECTION:
  $1/T \Longrightarrow -3dB$ for first order pole and
  $1/2T \Longrightarrow$-0.97

$1/10T$	$1/2T$	$1/T$	$2/T$	$10/T$
5.7	26	45	63.5	84.3

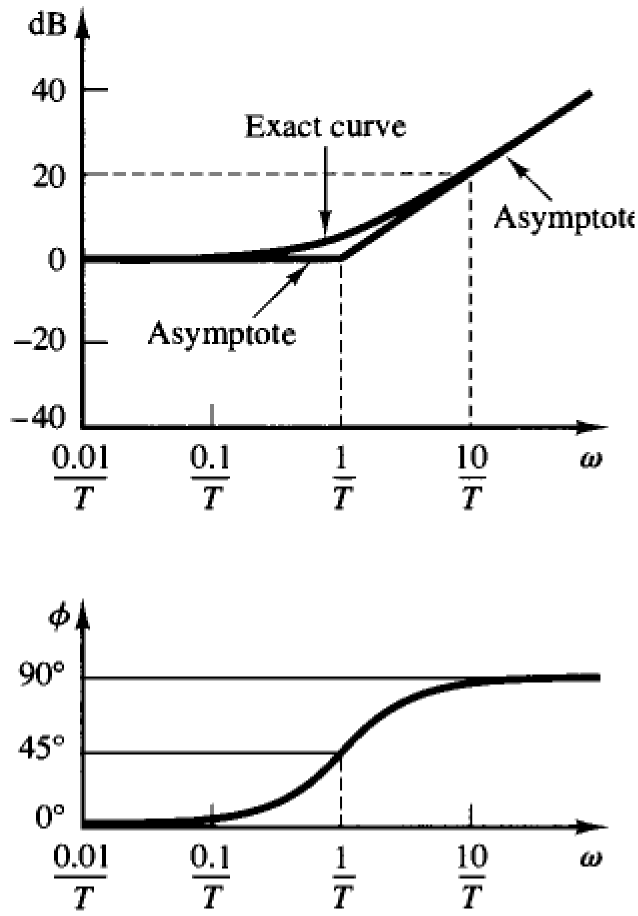


Figure 1: First order zero Bode plot

$$1/(1 + jwT) \Longrightarrow \begin{array}{c} |G| = -20log(\sqrt{1 + (\omega T)^2}) \\ PH(G) = -tan^{-1}(\omega T) \end{array}$$

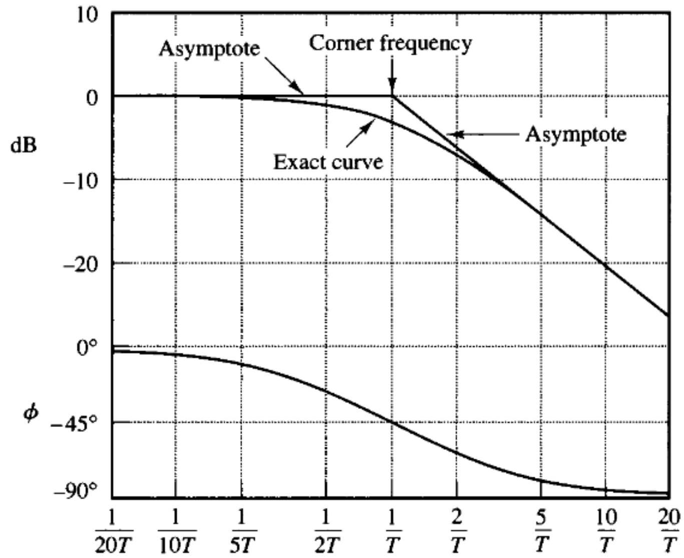


Figure 2: First Order Pole Bode Plot

Bode Plot Procedure

1. Put the transfer function in the standard form for different components
2. Determine the critical frequencies
3. draw the asymptotic log-magnitude and phase curves with proper slopes between corner frequencies
4. corrections may be then introduced at corner frequency

Example 1

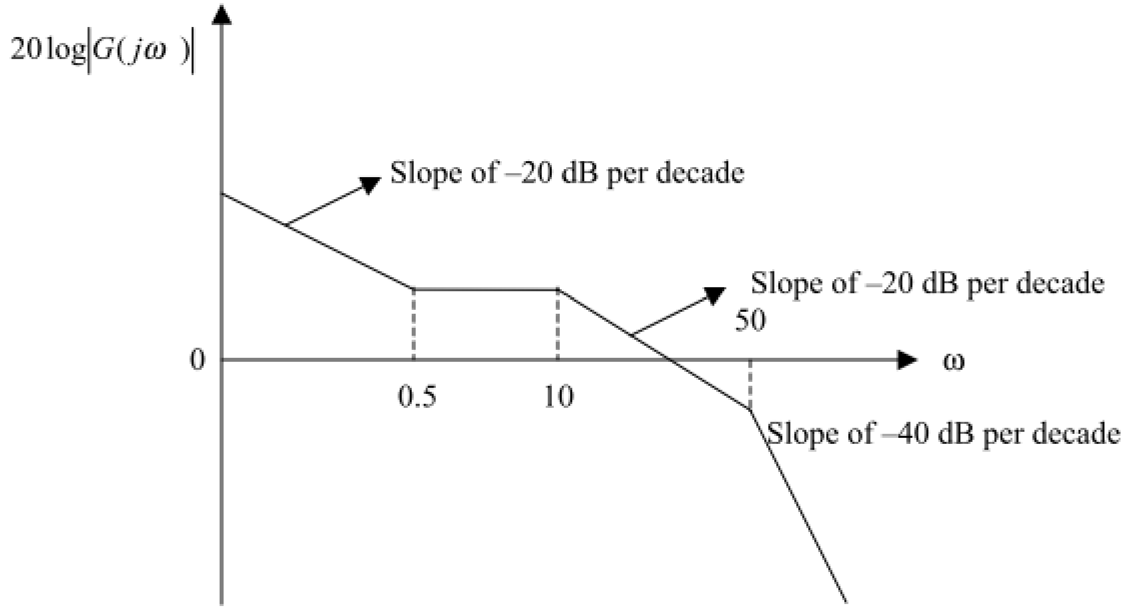
Plot the Bode diagram of the following open-loop transfer function

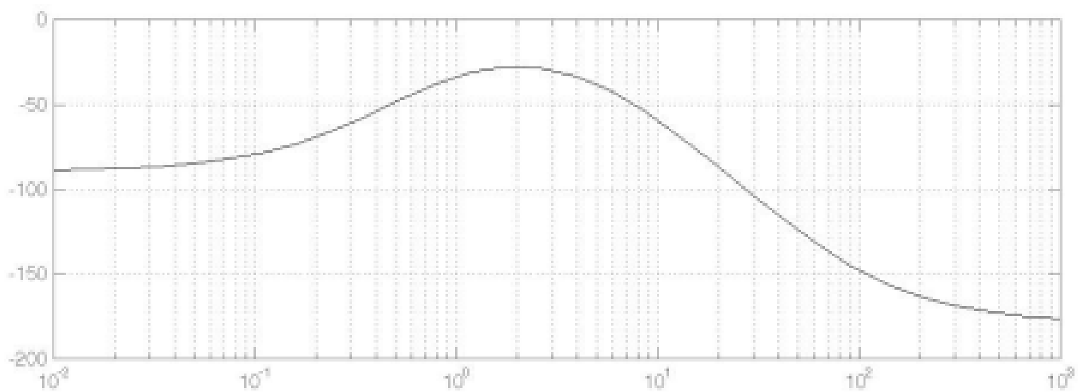
$$G(s) = \frac{2000(s + 0.5)}{s(s + 10)(s + 50)}$$

substituting $s = j\omega$ into G(s)

$$G(j\omega) = \frac{2(1 + \frac{j\omega}{0.5})}{j\omega(1 + \frac{j\omega}{10})(1 + \frac{j\omega}{50})}$$

Critical Frequencies: 0, 0.5, 10, 50





2.1.4 Second Order (Quadratic) Term $\left[1+2\zeta\left(j\omega/\omega_n\right)+\left(j\omega/\omega_n\right)^2\right]^{\pm 1}$

Second Order Factor $\frac{1}{1+2\zeta(j\omega/\omega_n)+(j\omega/\omega_n)^2}$

- $\zeta > 1 \Longrightarrow$ two first order terms
- $0 < \zeta < 1 \Longrightarrow$two complex-conjugate roots
- for very small values of ζ, the approximations may not be accurate because the magnitude and phase depends on both the damping factors in addition to the corner frequency
- $|G| = -20log\sqrt{(1-\left(\frac{w}{w_n}\right)^2)^2+(2\zeta\frac{\omega}{\omega_n})^2}$, $PH(G) = -tan^{-1}\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{w}{w_n}\right)^2}$
- Corner frequency: ω_n
- $\omega \lll \omega_n$: magnitude equals ZERO
  $\phi = tan^{-1}0 = 0$
- $\omega = \omega_n$, magnitude is highly dependent on ζ
  $\phi = -90deg$

- $\omega \ggg \omega_n$: magnitude drops 40dB/decade
  $-20log(\frac{\omega}{\omega_n})^2 = -40log(\frac{\omega}{\omega_n})$ dB
  $\phi = -180deg$

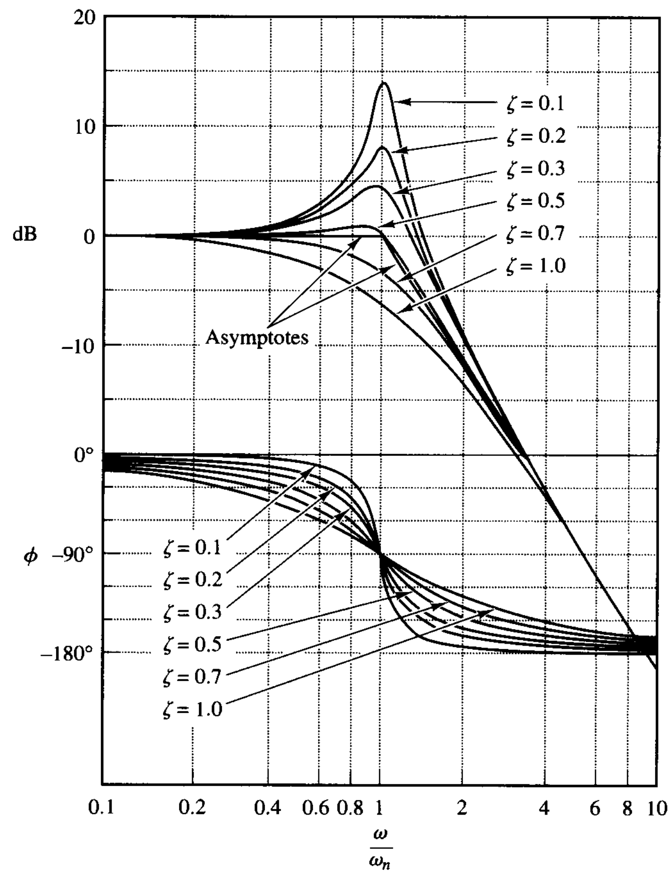


Figure 3: Second order Systems

Resonant Frequency ω_r and the resonant peak M_r

- The frequency at which the max peak occurs for 1/Quadratic factor
- ω_r can be obtained by differentiating the magnitude
- $\omega_r = \omega_n\sqrt{1-2\zeta^2}$ for $0 \le \zeta \le 0.707$, $M_r = |G| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$
- for larger ζ, no peak exist (check the figure). the system is oscillatory but well-damped :) $M_r = 1$
- $\zeta \to 0$, $\omega_r \to \omega_n$

Quadratic term Bode plot

- Magnitude is drawn using the asymptotes intersecting at the corner frequency ω_n.
  - A correction of $-20log2\zeta$ is applied at ω_n when required

- Phase shift
  - $1 > \zeta \geq 0.5$, we draw lines connecting $\omega_n/10$ and $10\,\omega_n$ with respective value of 0 and ± 180
  - $\zeta < 0.5$, we draw lines connecting $\omega_n/5^{\zeta}$ and $5^{\zeta}\omega_n$ with respective value of 0 and ± 180

Example 2

Plot the Bode diagram of the following open-loop transfer function

$$G(s) = \frac{20(s^2 + s + 0.5)}{s(s+1)(s+10)}$$

substituting $s = j\omega$ into G(s),

$$G(j\omega) = \frac{((j\frac{w}{\sqrt{0.5}})^2 + 1.414(j\frac{w}{\sqrt{0.5}}) + 1)}{j\omega(j\omega+1)(j0.1\omega+1)}$$

Critical frequencies: 0, 1, 10, $\omega_n = \sqrt{0.5} = 0.707$, $\zeta = 0.707$

2.2 Minimum and non-min-phase systems

- Min phase systems are systems without zeros or poles in the RHP.
- Illustrative Example: $G_1(j\omega) = \frac{1+j\omega T}{1+j\omega T_1}$ and $G_1(j\omega) = \frac{1-j\omega T}{1+j\omega T_1}$ and $0 < T < T_1$
  - both systems have the same magnitude but different phases as shown in Figure 4

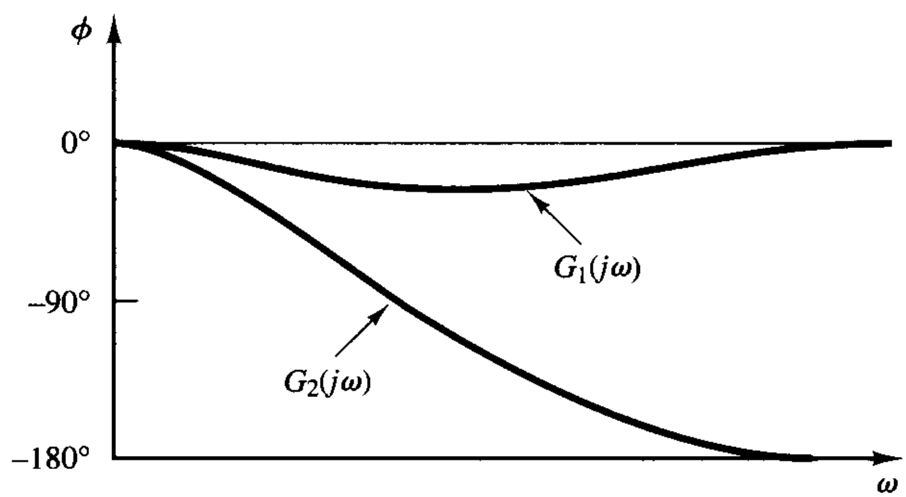


Figure 4: Phase angle [PH(G_1]=$-2tan^{-1}\omega T$

  - non min-phase systems are slow in response and such behavior is not required in most of the practical applications.

- Characteristics of min-phase systems

- Min phase systems have the min phase shift in comparison to all systems with the same magnitude factor
- Min-phase system can be completely represented using the magnitude curve. The phase curve is related to the magnitude by Hilbert transform.
- phase angle as $\omega \to \infty$, $\phi \to -90(q-p)$, where $q\, and\, p$ are the degrees of the denominator and the numerator.

- **Transport Lag** $G(j\omega) = e^{-j\omega T}$

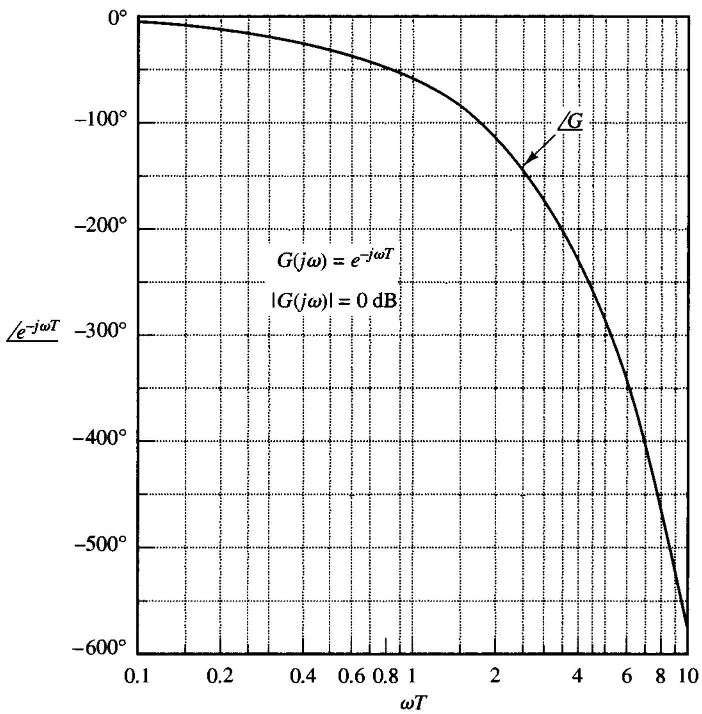


Figure 5: **Transport Lag Phase**

- Magnitude $|cos\omega T - jsin\omega T| = 1$
- Phase: $tan^{-1}tan\omega T = \omega T\, rad = -57.3\omega T\, deg$
- Linear change in the phase versus the frequency.

Example 3

Plot the Bode diagram for $G(s) = \frac{e^{-j\omega L}}{1+j\omega T}$

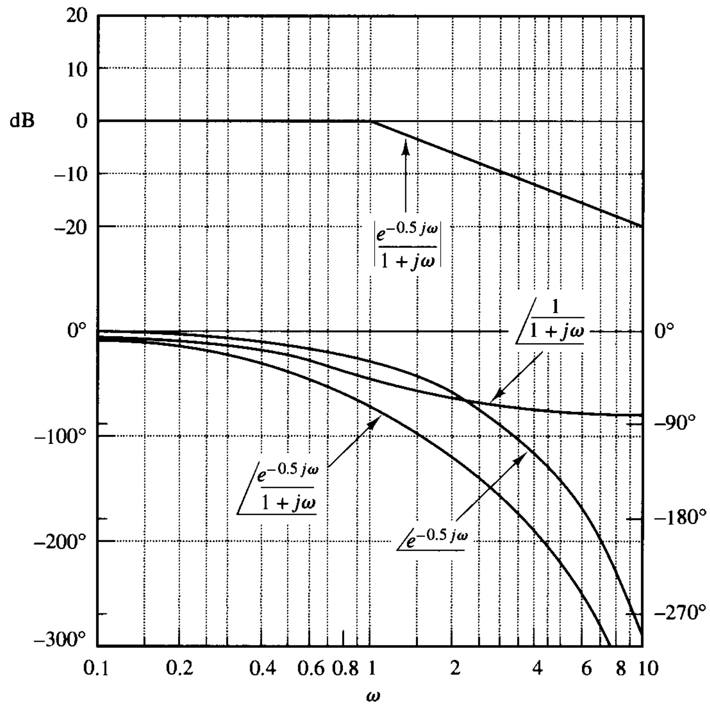


Figure 6: Example 3 (Transport Lag)

2.3 Plotting Bode Plot Using Matlab

```
w=logspace(-1,2);
num=[0 0 25];
den=[1 4 25];
bode(num,den,w);
```

3 Analyzing System Performance Using Bode Plots

3.1 Steady State Error Analysis Using Bode Plot

- **For Unity feedback system**, the static, position, and acceleration error constants describe the low-frequency behavior of type 0, type-1, and type-2 respectively
- Only ONE static errors is finite and significant depending on the system type

	step input r(t)=1	ramp input (r(t)=t)	parabolic input (r(t)=$t^2/2$)
Type-0	1/(K+1)	∞	∞
Type-1	0	1/K	∞
Type-2	0	0	1/K

Table 1: Steady State Error and System Gain

- The type of the system determines the slope of log-magnitude curve at low frequencies. Hence, the low-frequency behavior can be used to determine the static error constants
- For Type-0 Systems, the horizontal line at low frequencies equals $20logK_p$

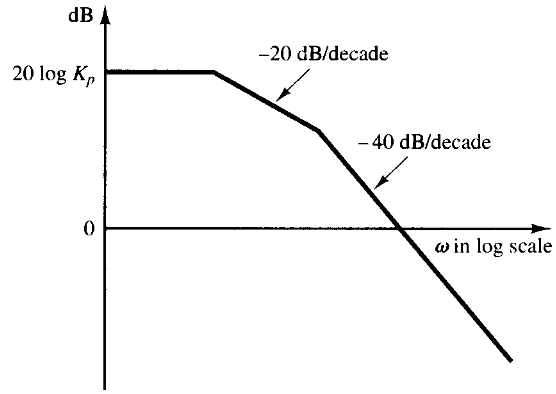


Figure 7: Static Error Constant and System Type 0

- **For Type-1 systems**, $\omega = 1 \rightarrow 20logK_v$. Also, the intersection frequency of the -20dB/dec line and the horizontal line equals K_v, why?

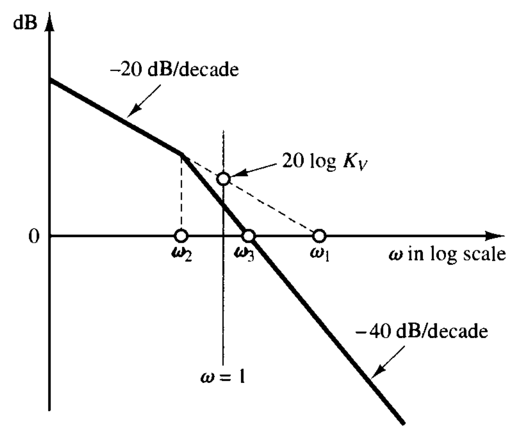


Figure 8: Static Error Constant and System Type 1

- **For Type-2 Systems**, the intersection frequency of the -40dB/dec line and the horizontal line equals K_a, why?

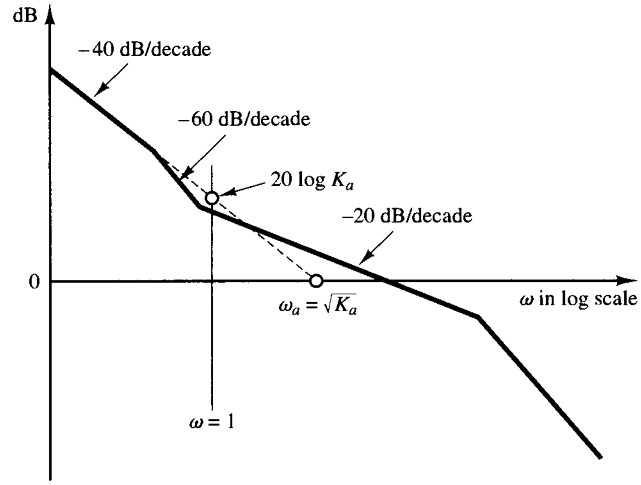


Figure 9: Static Error Constant and System Type 2

3.2 Analyzing System Stability Using Bode Plot

- Gain cross-over frequency (ω_{gc}):
  - is the frequency where the amplitude ratio is 1, or when log modulus is equal to 0
  - $|G(j\omega_{gc})H(j\omega_{gc})| = 1$
- Phase cross-over frequency (ω_{pc}):
  - is the frequency where phase shift is equal to -180^o.
  - $\angle G(j\omega_{pc})H(j\omega_{pc}) = 180^o$

3.2.1 Stability Criteria (Is my system stable?)

- If at the phase crossover frequency, the corresponding log modulus of G($j\omega_{pc}$) is less than 0 dB, then the feedback system is stable
- $\omega_{pc} > \omega_{gc} \Longrightarrow$ Stable system

3.2.2 Relative Stability

- The relative stability measures the system immunity to changes in the input due to noise or sudden changes.
- Relative stability also indicates how close the roots of the closed-loop characteristic equation 1+G(s)H(s) are to the jw axis.
- Relative stability is usually expressed in terms of gain and phase margin measure.
- Gain Margin
  - is the amount of change in the value of the gain of the transfer function, from its present value, to that value that will make the magnitude Bode plot pass through the 0 db at the same frequency where the phase is -180 degrees.is

- the inverse of transfer function magnitude at ω_{pc}.

- Phase Margin
  - is the amount of pure phase shift (no change in magnitude) that will make the phase shift of G(jw)H(jw) equal to -180 degrees at the same frequency where the magnitude is 0 db (1 in absolute value).
  - Let ϑ=PH(G($j\omega_{gc}$)) then the phase margin is given by $PM = 180^o + \vartheta$

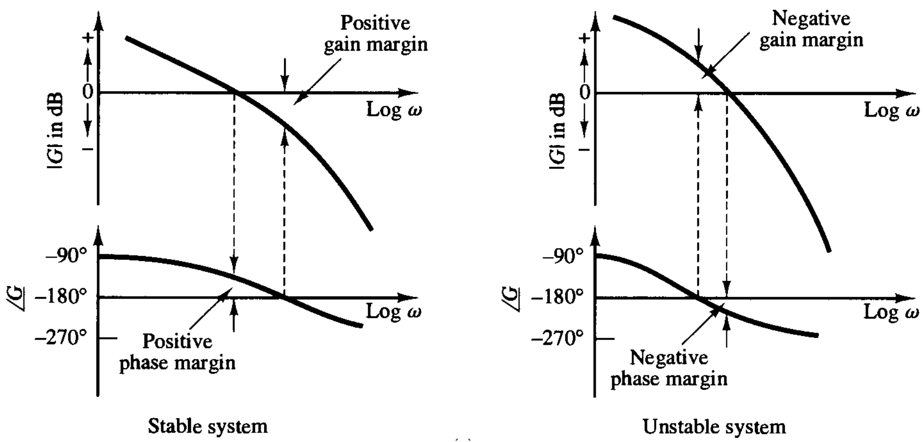
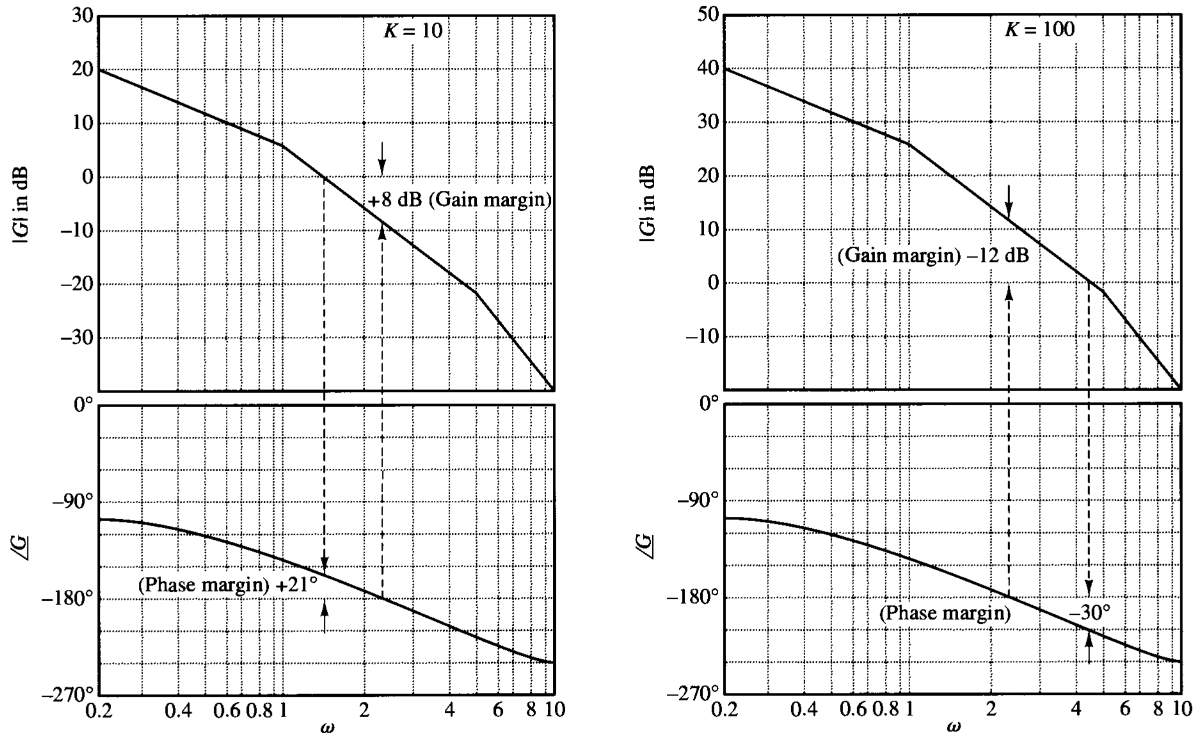


Figure 10: Bode Relative Stability Analysis

- The gain margin of first and second order systems is infinite, why? Hence these systems, ideally, are always stable.
- Any of the aforementioned margins alone is not a sufficient indication for relative stability
- For a satisfactory performance, a phase margin between 30 and 60 and a gain margin greater than 6dB are required.

Example

Consider a unity feedback system with $G(s) = \frac{K}{s(s+1)(s+5)}$ for K=10, 100
Solution
K=10, we have Phase Margin=21 and Gain Margin = 8dB
K=10, we have Phase Margin=-30 and Gain Margin = -12dB


K = 10
|G| in dB
+8 dB (Gain margin)
(Phase margin) +21°
∠G
ω
K = 100
|G| in dB
(Gain margin) −12 dB
(Phase margin)
−30°
∠G
ω