

Solution of Sheet 7  
Control Systems Design in State Space

[1]

$$|sI - A| = \begin{vmatrix} s+1 & 0 & -1 \\ -1 & s+2 & 0 \\ 0 & 0 & s+3 \end{vmatrix} = s^3 + 6s^2 + 11s + 6 = 0$$

$$W = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

a)

$$M = \begin{bmatrix} 0 & 1 & -4 \\ 0 & 0 & 1 \\ 1 & -3 & 9 \end{bmatrix}$$

$|M| \neq 0, \therefore$  The system is controllable

$$T_c = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & 3 & 1 \end{bmatrix}, \quad T_c^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -2 & 0 \\ -3 & 4 & 1 \end{bmatrix}$$

$$A_c = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C_c = [3 \quad 1 \quad 0]$$

b)

$$N = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 4 & -3 \end{bmatrix}$$

$|N| = 0, \therefore$  The system is unobservable

[2]

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$|M| = 0, \therefore$  The system is uncontrollable

[3]

$$|sI - A| = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s+3 \end{vmatrix} = s^3 + 3s^2 = 0$$

$$W = \begin{bmatrix} 0 & 3 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & -3 \\ 1 & -3 & 9 \end{bmatrix}$$

$|M| \neq 0, \therefore$  The system is controllable

$$T_c = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_c^{-1} = \frac{1}{16} \begin{bmatrix} 4 & -1 & 1 \\ 0 & 4 & -4 \\ 0 & 0 & 16 \end{bmatrix}$$

$$(s + 2 - j4)(s + 2 + j4)(s + 10) = s^3 + 14s^2 + 60s + 200$$

$$K = [200 \quad 60 \quad 11]T_c^{-1} = [50 \quad 2.5 \quad 8.5]$$

[4]

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3 + 6s^2 + 11s + 6}$$

$$\ddot{y} + 6\dot{y} + 11y = 10u$$

$$\dot{x}_3 + 6x_3 + 11x_2 + 6x_1 = 10u$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]$$

$$(s + 2 - j2\sqrt{3})(s + 2 + j2\sqrt{3})(s + 10) = s^3 + 14s^2 + 56s + 160$$

$$K = \frac{1}{10} \times [154 \quad 45 \quad 8] = [15.4 \quad 4.5 \quad 0.8]$$

[5]

$$|sI - A| = \begin{vmatrix} s+1 & -1 \\ -1 & s+2 \end{vmatrix} = s^2 + 3s + 1 = 0$$

$$W = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$|N| \neq 0, \therefore$  The system is observable

$$T_o = WN = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad T_o^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

$$(s + 5)(s + 5) = s^2 + 10s + 25$$

$$K_e = T_o^{-1} \begin{bmatrix} 24 \\ 7 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \end{bmatrix}$$

[6]

$$|sI - A| = 0 = s^3 + 6s + 5$$

$$W = \begin{bmatrix} 6 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|N| \neq 0, \therefore$  The system is observable

$$T_o = \begin{bmatrix} 6 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T_o^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -6 \end{bmatrix}$$

$$(s + 10)^2(s + 15) = s^3 + 35s^2 + 400s + 1500$$

$$K_e = T_o^{-1} \begin{bmatrix} 1495 \\ 394 \\ 35 \end{bmatrix} = \begin{bmatrix} 35 \\ 394 \\ 1285 \end{bmatrix}$$

[7]

$$|sI - A| = 0 = s^3 + 6s^2 + 11s + 6$$

$$W = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$|M| \neq 0, \therefore$  The system is controllable

$$T_c = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_c^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(s + 1 - j)(s + 1 + j)(s + 5) = s^3 + 7s^2 + 12s + 10$$

$$K = [4 \quad 1 \quad 1]T_c^{-1} = [4 \quad 1 \quad 1]$$

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$|N| \neq 0, \therefore$  The system is observable

$$T_o = \begin{bmatrix} 11 & 6 & 1 \\ 6 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad T_o^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

$$(s + 6)^3 = s^3 + 18s^2 + 108s + 216$$

$$K_e = T_o^{-1} \begin{bmatrix} 210 \\ 79 \\ 12 \end{bmatrix} = \begin{bmatrix} 12 \\ 25 \\ -72 \end{bmatrix}$$