CAIRO UNIVERSITY
ELECTRONICS & COMMUNICATIONS DEP.
CONTROL ENGINEERING

FACULTY OF ENGINEERING
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SHEET 7

Control Systems Design in State Space

[1] Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Transform the system equation into:

a) Controllable canonical form.
b) Observable canonical form.

[2] Consider the system defined by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

Show that this system cannot be stabilized by the state-feedback control $u = -\boldsymbol{K}\boldsymbol{x}$, whatever the matrix $\boldsymbol{K}$ is chosen.

[3] Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

By using the state-feedback control $u = -\boldsymbol{K}\boldsymbol{x}$, it is desired to have the closed-loop poles at

$$s = -2 \pm j4, \qquad s = -10.$$

Determine the state-feedback gain matrix $\boldsymbol{K}$.

[4] A regulator system has plant

$$\frac{Y(s)}{U(s)} = \frac{10}{(s+1)(s+2)(s+3)}$$

Define state variables as

$$x_1 = y, \quad x_2 = \dot{x}_1, \quad x_3 = \dot{x}_2.$$

By use of the state-feedback control $u = -\boldsymbol{K}\boldsymbol{x}$, it is desired to place the closed-loop poles at

$$s = -2 \pm j2\sqrt{3}, \quad s = -10.$$

Determine the necessary state-feedback matrix $\boldsymbol{K}$.

[5] Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$$
$$y = \boldsymbol{C}\boldsymbol{x}$$

where

$$\boldsymbol{A} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Design a full-order state observer. The desired observer poles are

$$s = -5, \quad s = -5$$

[6] Consider the system defined by

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Design a full-order state observer. Assuming that the desired poles for the observer are located at

$$s = -10, \quad s = -10, \quad s = -15$$

[7] Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Design a regulator system by the ***pole placement with observer*** approach. Assume that the desired closed-loop poles for pole placement are located at

$$s = -1 \pm j, \qquad s = -5.$$

The desired observer poles are located at

$$s = -6, \qquad s = -6, \qquad s = -6.$$

Also obtain the transfer function of the observer controller.

Summary

- ***Some definitions***

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u$$
$$y = \boldsymbol{C}\boldsymbol{x} + Du$$

where

x: state vector (n-vector)

y: output signal (scalar)

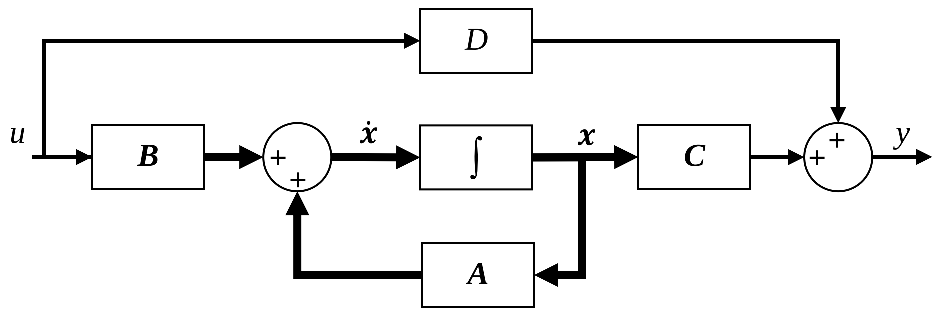
u: control signal (scalar)

A: n×n constant matrix

B: n×1 constant matrix

C: 1×n constant matrix

D: constant (scalar)



$$\boldsymbol{M} = [\boldsymbol{B}|\boldsymbol{AB}|\cdots|\boldsymbol{A}^{\boldsymbol{n-1}}\boldsymbol{B}]: \boldsymbol{Controllability\ matrix}$$

$$\boldsymbol{N} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \vdots \\ \boldsymbol{CA}^{\boldsymbol{n-1}} \end{bmatrix}: \boldsymbol{Observability\ matrix}$$

$$\boldsymbol{W} = \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & 1 \\ a_{n-2} & a_{n-3} & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ a_1 & 1 & \cdots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the $a_i's$ are coefficients of the characteristic polynomial

$$|s\boldsymbol{I} - \boldsymbol{A}| = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

- ***Controllable canonical form***

$$\boldsymbol{x_c} = \boldsymbol{T_c^{-1}x}$$

where

$\mathbf{x_c}$ is the state vector in the controllable canonical form

$\mathbf{T_c}$ is the transformation matrix

$$\boldsymbol{T_c} = \boldsymbol{MW}$$

$$\boldsymbol{A_c} = \boldsymbol{T_c^{-1}AT_c} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}$$

$$\boldsymbol{B_c} = \boldsymbol{T_c^{-1}B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{C_c} = \boldsymbol{CT_c}$$

- ***observable canonical form***

$$\boldsymbol{x_o} = \boldsymbol{T_o x}$$

where

x$_o$ is the state vector in the observable canonical form

T$_o$ is the transformation matrix

$$\boldsymbol{T_o} = \boldsymbol{WN}$$

$$\boldsymbol{A_o} = \boldsymbol{T_o A T_o^{-1}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}$$

$$\boldsymbol{B_o} = \boldsymbol{T_o B}, \qquad \boldsymbol{C_o} = \boldsymbol{CT_o^{-1}} = [0 \quad 0 \quad \cdots \quad 0 \quad 1]$$

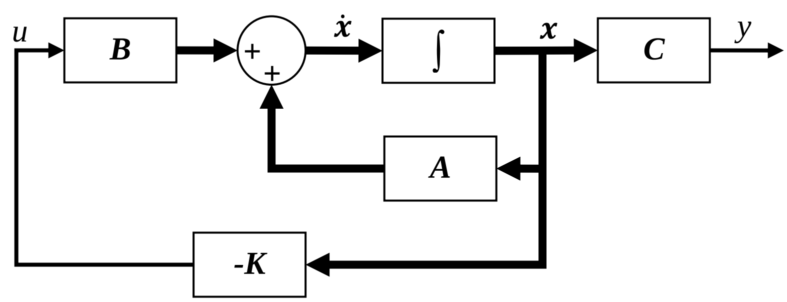
- ***State Feedback:***

The system uses the state feedback $u = -\boldsymbol{Kx}$. The target is to determine the state feedback gain matrix ***K***, to have the system poles at $\mu_1, \mu_2, \cdots, \mu_n$.

$$(s-\mu_1)(s-\mu_2)\cdots(s-\mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n = 0$$

$$\boldsymbol{K} = [\alpha_n - a_n | \alpha_{n-1} - a_{n-1} | \cdots | \alpha_2 - a_2 | \alpha_1 - a_1]\boldsymbol{T_c^{-1}}$$

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x}$$



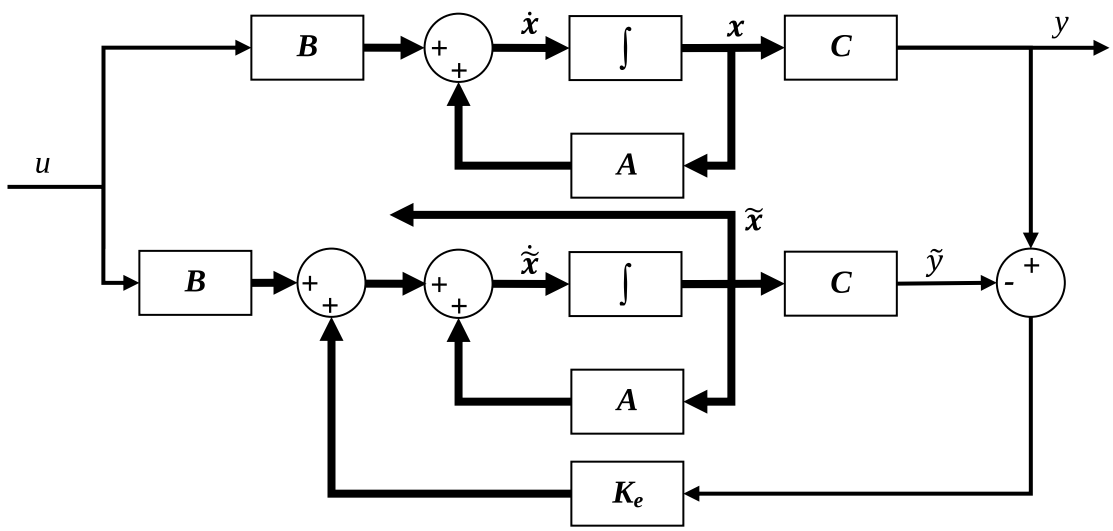
- ***Full-Order State Observer:***

The system uses the state observer to estimate the state variables and the error in the states $\boldsymbol{e} = \boldsymbol{x} - \tilde{\boldsymbol{x}}$, having dynamic behavior $\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{K_e C})\boldsymbol{e}$. The target is to determine the observer gain matrix $\boldsymbol{K_e}$, to have the observer poles at $\mu_1, \mu_2, \cdots, \mu_n$.

$$(s - \mu_1)(s - \mu_2) \cdots (s - \mu_n) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1} s + \alpha_n = 0$$

$$\boldsymbol{K_e} = \boldsymbol{T_o^{-1}} \begin{bmatrix} \alpha_n - a_n \\ \alpha_{n-1} - a_{n-1} \\ \vdots \\ \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix}$$

$$\dot{\tilde{\boldsymbol{x}}} = (\boldsymbol{A} - \boldsymbol{K_e C})\tilde{\boldsymbol{x}} + \boldsymbol{B}u + \boldsymbol{K_e} y$$



- ***Pole placement with observer:***

The feedback and observer are designed separately