Topic 4

Linear Wire and Small Circular Loop Antennas

Tamer Abuelfadl

Electronics and Electrical Communications Department
Faculty of Engineering
Cairo University

Linear Wire Antennas and Small Circular Loop Antenna

Outline

Infinitesimal Dipole

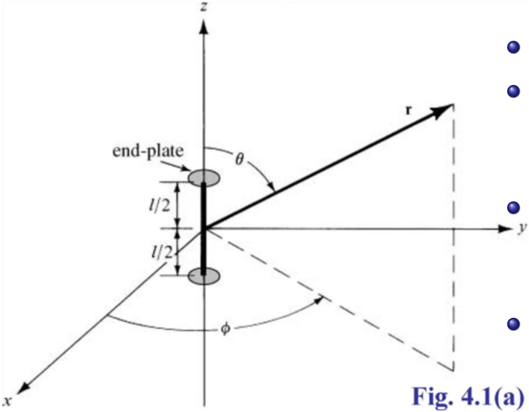


Fig. 4.1(a)

- $l \ll \lambda$ $(l \leq \lambda/50)$
- End plates are to maintain uniform current, however they are very small to affect radiation!
- Not very practical, however they are considered as a basic building blocks for complex structures.
- $\mathbf{I}_e(z') = \hat{\mathbf{a}}_z I_0$

$$\mathbf{A}(r,\theta,\phi) = \frac{\mu}{4\pi}\int_C \mathbf{I}_e \frac{e^{-jkR}}{R} dl' = \hat{\mathbf{a}}_z \frac{\mu I_0 l}{4\pi r} e^{-jkr}$$

Infinitesimal Dipole

- Transformation from rectangular spherical components,

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$A_r = \frac{\mu I_0 l e^{-jkr}}{4\pi r}\cos\theta, \qquad A_\theta = -\frac{\mu I_0 l e^{-jkr}}{4\pi r}\sin\theta$$

- Magnetic field

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} = \hat{\mathbf{a}}_\phi \frac{1}{\mu r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right] = j\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta\left[1+\frac{1}{jkr}\right]\hat{\mathbf{a}}_\phi$$

- Electric field

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla\times\mathbf{H}$$

Infinitesimal Dipole

$$\mathbf{E} = j\eta \frac{kI_0 l}{4\pi r} e^{-jkr} \left[2\cos\theta \left(\frac{1}{jkr} - \frac{1}{k^2 r^2} \right) \hat{\mathbf{a}}_r + \sin\theta \left(1 + \frac{1}{jkr} - \frac{1}{k^2 r^2} \right) \hat{\mathbf{a}}_\theta \right]$$

$$\mathbf{H} = j \frac{kI_0 l e^{-jkr}}{4\pi r} \sin\theta \left[1 + \frac{1}{jkr} \right] \hat{\mathbf{a}}_\phi$$

- Radiation Power Density,

$$\mathbf{W} = \frac{1}{2} (\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} \left(\hat{\mathbf{a}}_r E_\theta H_\phi^* - \hat{\mathbf{a}}_\theta E_r H_\phi^* \right)$$

- Radiated Power,

$$\begin{aligned} P_{\text{rad}} &= \Re \left\{ \oiint_S \mathbf{W} \cdot ds \right\} = \eta \frac{k^2 I_0^2 l^2}{32\pi^2} \int_0^{2\pi} d\phi \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta \\ &= \frac{\eta\pi}{4} \left| I_0 \frac{l}{\lambda} \right|^2 \left[-\cos\theta + \frac{\cos^3\theta}{3} \right]_0^{\pi} = \frac{\eta\pi}{3} \left| I_0 \frac{l}{\lambda} \right|^2 \end{aligned}$$

- Radiation Resistance

$$P_{\text{rad}} = \frac{\eta\pi}{3} \left| I_0 \frac{l}{\lambda} \right|^2 = \frac{1}{2} |I_0|^2 R_r \quad \Longrightarrow \quad \boxed{R_r = 80\pi^2 \left(\frac{l}{\lambda} \right)^2}$$

Radiation Regions

- Near field region $kr \ll 1$,

$$\mathbf{E} = -j\eta \frac{I_0 l}{4\pi k r^3} e^{-jkr} \left[2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta\right]$$

$$\mathbf{H} = \frac{I_0 l e^{-jkr}}{4\pi r^2} \sin\theta \hat{\mathbf{a}}_\phi$$

- Intermediate-field (Fresnel) region $kr > 1$,

$$\mathbf{E} = j\eta \frac{k I_0 l}{4\pi r} e^{-jkr} \left[\frac{2}{jkr} \cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta\right]$$

$$\mathbf{H} = j \frac{k I_0 l e^{-jkr}}{4\pi r} \sin\theta \hat{\mathbf{a}}_\phi$$

- Far-field (Fraunhover) region $kr \gg 1$,

$$\mathbf{E} = j\eta \frac{k I_0 l \sin\theta}{4\pi r} e^{-jkr} \hat{\mathbf{a}}_\theta, \qquad \mathbf{H} = j \frac{k I_0 l e^{-jkr}}{4\pi r} \sin\theta \hat{\mathbf{a}}_\phi$$

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{a}}_r \times \mathbf{E}$$

Directivity

$$W_{\text{rad}} = \frac{|E_\theta|^2}{2\eta}$$

$$U = r^2 W_{\text{rad}} = r^2 \frac{|E_\theta|^2}{2\eta} = \eta \frac{k^2 I_0^2 l^2}{32\pi^2} \sin^2\theta = \frac{\eta}{8} \left| I_0 \frac{l}{\lambda} \right|^2 \sin^2\theta$$

$$P_{\text{rad}} = \frac{\eta\pi}{3} \left| I_0 \frac{l}{\lambda} \right|^2$$

$$D = \frac{4\pi U}{P_{\text{rad}}} = 1.5 \sin^2\theta$$

Outline

Small Dipole

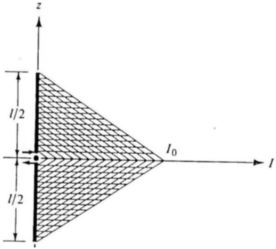
$$\mathbf{I}_e\left(z'\right) = \begin{cases} \hat{\mathbf{a}}_z I_0\left(1-\frac{2z'}{l}\right) & 0 \le z' \le l/2 \\ \hat{\mathbf{a}}_z I_0\left(1+\frac{2z'}{l}\right) & -l/2 \le z' \le 0 \end{cases}$$

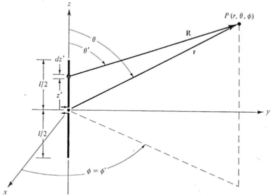
$$\mathbf{A} = \hat{\mathbf{a}}_z \frac{1}{2}\left[\frac{\mu I_0 l e^{-jkr}}{4\pi r}\right]$$

$$\mathbf{E} = j\eta \frac{k I_0\left(l/2\right)e^{-kr}}{4\pi r}\sin\theta \hat{\mathbf{a}}_\theta$$

$$\mathbf{H} = j\frac{k I_0\left(l/2\right)e^{-kr}}{4\pi r}\sin\theta \hat{\mathbf{a}}_\phi$$

- Radiation resistance $R_r = 80\pi^2\left(\frac{l/2}{\lambda}\right)^2$

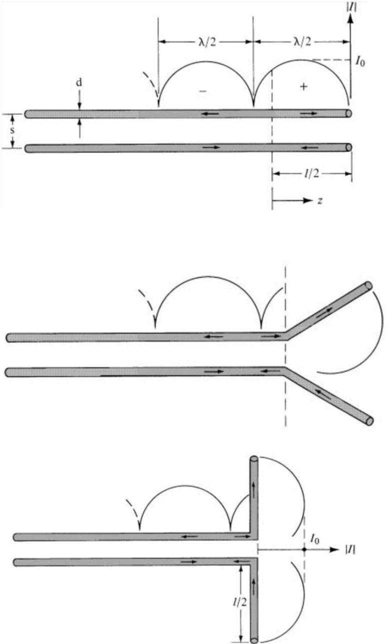


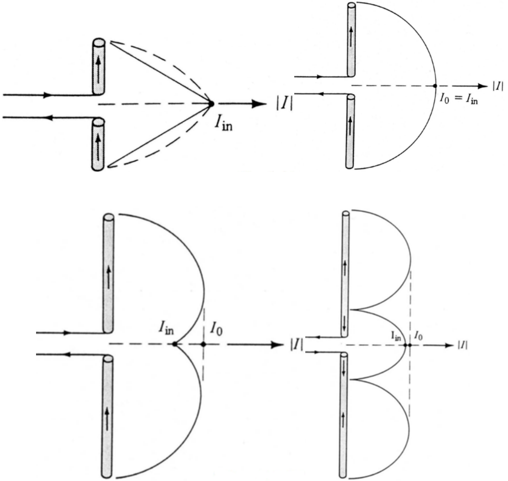


Small dipole of length l is equivalent to infinitesimal dipole of length $l/2$.

Outline

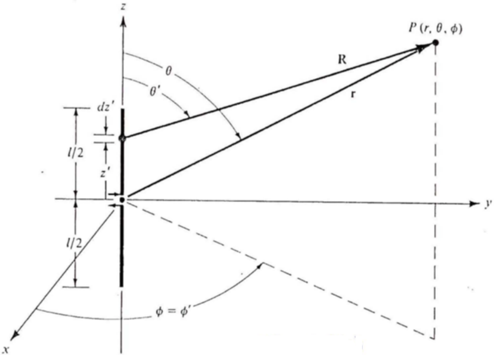
Finite Length Dipole





$$\mathbf{I}_e\left(z'\right) = \begin{cases} \hat{\mathbf{a}}_z I_0 \sin\left[k\left(\frac{l}{2} - z'\right)\right], & 0 \leq z' \leq \frac{l}{2} \\ \hat{\mathbf{a}}_z I_0 \sin\left[k\left(\frac{l}{2} + z'\right)\right], & -\frac{l}{2} \leq z' \leq 0 \end{cases}$$

Finite Length Dipole

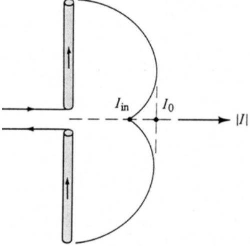


$$\mathbf{E} \backsimeq -j\omega \left[A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi \right],$$

$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_C \mathbf{I}_e(r',\theta',\phi') e^{j\mathbf{k}\cdot\mathbf{r}'} dl'$$

$$\mathbf{A} = \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_{-l/2}^{l/2} I_e(z') e^{jkz'\cos\theta} dz' \hat{\mathbf{a}}_z$$

$$\mathbf{E} = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \int_{-l/2}^{l/2} I_e(z') e^{jkz'\cos\theta} dz' \hat{\mathbf{a}}_\theta$$



$$\mathbf{I}_e\left(z'\right) = \begin{cases} \hat{\mathbf{a}}_z I_0 \sin\left[k\left(\frac{l}{2} - z'\right)\right], & 0 \le z' \le \frac{l}{2} \\ \hat{\mathbf{a}}_z I_0 \sin\left[k\left(\frac{l}{2} + z'\right)\right], & -\frac{l}{2} \le z' \le 0 \end{cases}$$

$$\mathbf{E} = j\eta \frac{ke^{-jkr}}{4\pi r} \sin\theta \int_{-l/2}^{l/2} I_e\left(z'\right) e^{jkz'\cos\theta} dz' \hat{\mathbf{a}}_\theta$$

$$\begin{aligned} \mathbf{E} \;=\; & j\eta \frac{kI_0 e^{-jkr}}{4\pi r} \sin\theta \left\{ \int_{-l/2}^{0} \sin\left[k\left(\frac{l}{2} + z'\right)\right] e^{jkz'\cos\theta} dz' \right. \\ & \left. + \int_{0}^{l/2} \sin\left[k\left(\frac{l}{2} - z'\right)\right] e^{jkz'\cos\theta} dz' \right\} \hat{\mathbf{a}}_\theta \end{aligned}$$

The integrals can be evaluated using

$$\int e^{\alpha x} \sin\left(\beta x + \gamma\right) dx = \frac{e^{\alpha x}}{\alpha^2 + \beta^2} \left[\alpha \sin\left(\beta x + \gamma\right) - \beta \cos\left(\beta x + \gamma\right)\right]$$

$$\mathbf{E} = j\eta \frac{I_0 e^{-jkr}}{2\pi r} \left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right) - \cos\left(\frac{kl}{2}\right)}{\sin\theta} \right] \hat{\mathbf{a}}_\theta$$

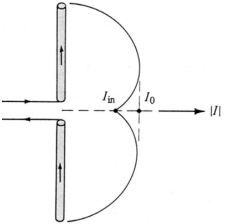
Radiation Intensity, and Radiation Resistance

$$U = \frac{r^2}{2\eta}\left|E_\theta\right|^2 = \frac{\eta\left|I_0\right|^2}{8\pi^2}\left[\frac{\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)}{\sin\theta}\right]^2$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U\sin\theta\, d\theta\, d\phi = \frac{\eta\left|I_0\right|^2}{4\pi}\int_0^{\pi}\frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta}\, d\theta$$

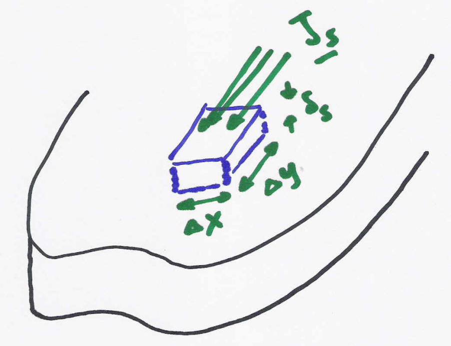
$$P_{\text{rad}} = \frac{1}{2}\left|I_{in}\right|^2 R_r = \frac{1}{2}\left|I_0\right|^2 \sin^2\left(kl/2\right) R_r$$

$$R_r = \frac{1}{\sin^2\left(kl/2\right)}\frac{\eta}{2\pi}\int_0^{\pi}\frac{\left[\cos\left(\frac{kl}{2}\cos\theta\right)-\cos\left(\frac{kl}{2}\right)\right]^2}{\sin\theta}\, d\theta$$



Outline

Conductor Losses



Skin depth:

$$\delta_s = 1/\sqrt{\pi f \mu \sigma}$$

- Resistance of the segment with dimensions (Δx, Δy, δ_s):

$$R = \frac{\Delta y}{\sigma \delta_s \Delta x}$$

- Current flowing through length Δx: $\Delta I_s = |\mathbf{J}_s| \Delta x$

Conductor Losses

$$R = \frac{\Delta y}{\sigma \delta_s \Delta x}$$

- Current flowing through length Δx: $\Delta I_s = |\mathbf{J}_s| \Delta x$

- Power loss in the area $\Delta x \Delta y$,

$$\Delta P_{loss} = \frac{1}{2} |\Delta I_s|^2 R = \frac{1}{2} |\mathbf{J}_s|^2 \frac{1}{\sigma \delta_s} \Delta x \Delta y$$

- Total Power loss in a conductor area S,

$$\boxed{P_{loss} = \frac{1}{2} \int_S |\mathbf{J}_s|^2 R_s ds, \quad \text{where } R_s = \frac{1}{\sigma \delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}}$$

Conductor Losses

$$\boxed{P_{loss} = \frac{1}{2}\int_S |\mathbf{J}_s|^2 R_s ds, \quad \text{where } R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}}$$

R_s is called the *surface resistance* of the conductor.

- The surface current can be obtained using the approximation of a perfect conductor boundary conditions,

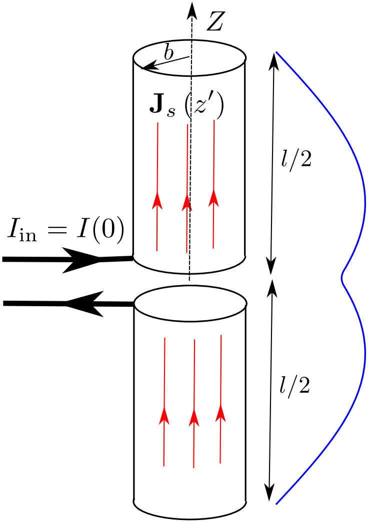
$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H} \quad \Longrightarrow \quad |\mathbf{J}_s| = |\mathbf{H}|$$

  where $\mathbf{n}$ is the normal unit vector on the conductor.

  - Total Power loss in a conductor area S,

$$\boxed{P_{loss} = \frac{1}{2}\int_S |\mathbf{H}|^2 R_s ds, \quad \text{where } R_s = \frac{1}{\sigma\delta_s} = \sqrt{\frac{\pi f \mu}{\sigma}}}$$

Loss resistance



$$P_{\text{loss}} = \frac{1}{2}\int_S |\mathbf{J}_s|^2 R_s ds$$
$$= \frac{1}{2}\int_{-l/2}^{l/2} \frac{|I_e(z')|^2}{(2\pi b)^2} R_s (2\pi b)\, dz'$$

$$P_{\text{loss}} = \frac{1}{2}\int_{-l/2}^{l/2} \frac{|I_e(z')|^2}{2\pi b} R_s dz' = \frac{1}{2}|I_e(0)|^2 R_L$$

$$\boxed{R_L = \frac{R_s}{2\pi b}\int_{-l/2}^{l/2} \left|\frac{I_e(z')}{I_e(0)}\right|^2 dz'}$$

Half-wavelength ($\lambda/2$) dipole

$$U = \frac{r^2}{2\eta}|E_\theta|^2 = \frac{\eta|I_0|^2}{8\pi^2}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2$$

$$P_{\text{rad}} = \int_0^{2\pi}\int_0^{\pi} U\sin\theta\, d\theta\, d\phi = \frac{\eta|I_0|^2}{4\pi}\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}d\theta$$

$$P_{\text{rad}} = \frac{1}{2}|I_0|^2 R_r$$

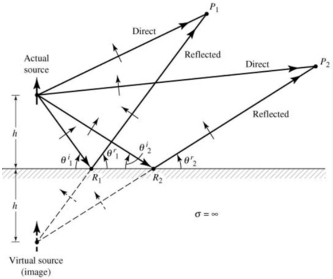
$$R_r = \frac{\eta}{2\pi}\int_0^{\pi}\frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}d\theta = 73\,\Omega$$

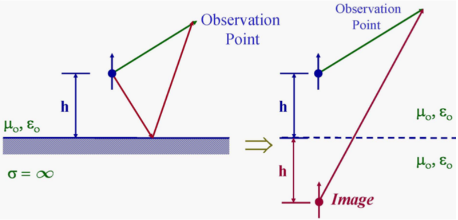
$$D = \frac{4\pi U}{P_{\text{rad}}} = \frac{\eta}{\pi R_r}\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2 = 1.64\left[\frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}\right]^2$$

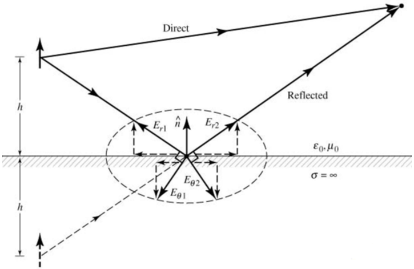
Outline

Linear Elements Near or on Infinite Conductor

Image Theory





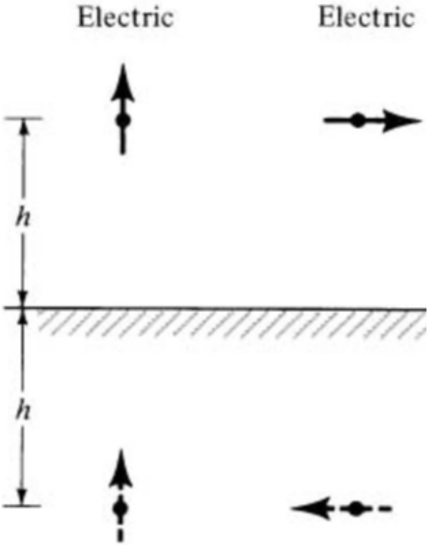


$$\mathbf{E} = \frac{j\eta k I_0 l e^{-jkr}}{4\pi r}\left[\cos\theta\left(\frac{2}{jkr} - \frac{1}{k^2r^2}\right)\hat{\mathbf{a}}_r + \sin\theta\left(1 + \frac{1}{jkr} - \frac{1}{k^2r^2}\right)\hat{\mathbf{a}}_\theta\right]$$

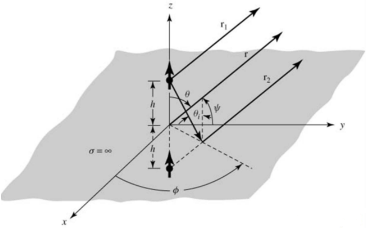
$$\mathbf{H} = j\frac{kI_0 l e^{-jkr}}{4\pi r}\sin\theta\left[1 + \frac{1}{jkr}\right]\hat{\mathbf{a}}_\phi$$

Linear Elements Near or on Infinite Conductor

Image Theory



Vertical Infinitesimal Dipole Above Ground Plane



$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_C \mathbf{I}_e e^{j\mathbf{k}\cdot\mathbf{r}'}\,dl'$$

$$\mathbf{H} \backsimeq \frac{\hat{\mathbf{a}}_r \times \mathbf{E}}{\eta}, \quad \mathbf{E} \backsimeq -j\omega\left[A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi\right]$$

$$\mathbf{E} = \begin{cases} j\eta \frac{kI_0 l e^{-kr}}{4\pi r}\sin\theta\,[2\cos(kh\cos\theta)]\,\hat{\mathbf{a}}_\theta & z \geq 0 \\ \mathbf{0} & z < 0 \end{cases}$$

$$U = \begin{cases} \frac{\eta}{2}\left|\frac{I_0 l}{\lambda}\right|^2 \sin^2\theta\cos^2(kh\cos\theta) & 0 < \theta \leq \pi/2 \\ 0 & \pi/2 < \theta \leq \pi \end{cases}$$

$$\begin{aligned} P_{\text{rad}} &= \int U d\Omega = \pi\eta\left|\frac{I_0 l}{\lambda}\right|^2 \int_0^{\pi/2} \sin^3\theta\cos^2(kh\cos\theta)\,d\theta \\ &= \pi\eta\left|\frac{I_0 l}{\lambda}\right|^2 \int_0^1 (1-x^2)\cos^2(khx)\,dx \end{aligned}$$

Vertical Infinitesimal Dipole Above Ground Plane

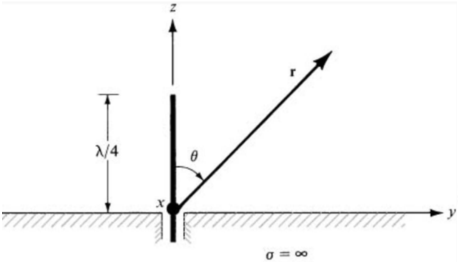
$$P_{\text{rad}} = \pi\eta \left|\frac{I_0 l}{\lambda}\right|^2 \left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]$$

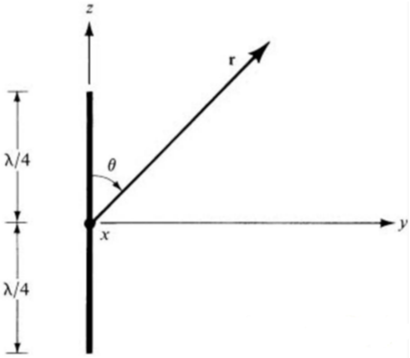
$$U_{\max} = \frac{\eta}{2}\left|\frac{I_0 l}{\lambda}\right|^2 \quad \Longrightarrow \quad \boxed{D_0 = \frac{4\pi U_{\max}}{P_{\text{rad}}} = \frac{2}{\left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]}}$$

$$\boxed{R_r = 2\pi\eta \left(\frac{l}{\lambda}\right)^2 \left[\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right]}$$

- $kh \to \infty$, $R_r = 80\pi^2 (l/\lambda)^2$, R_r is identical to *isolated Infinitesimal dipole*.
- $kh \to 0$, $D_0 = 3$, $R_r = 160\pi^2 (l/\lambda)^2$, D_0 and R_r are twice the *isolated Infinitesimal dipole*.

$\lambda/4$ Monopole on Infinite Electric Conductor





The radiation fields are identical in the upper half plane ($z > 0$). However the total radiated for the monopole is half its value for the dipole.

$$P_{\text{rad (m)}} = \frac{1}{2} P_{\text{rad (d)}}$$

$$D_m = 2D_d, \qquad R_{r\text{ (m)}} = \frac{1}{2} R_{r\text{ (d)}}$$

Outline

Angle Between Two Directions in Space

- The unit vector describing the the direction (θ, ϕ),

$$\hat{\mathbf{a}}_r = \sin\theta\cos\phi\,\hat{\mathbf{a}}_x + \sin\theta\sin\phi\,\hat{\mathbf{a}}_y + \cos\theta\,\hat{\mathbf{a}}_z$$

- The unit vector describing the the direction (θ', ϕ'),

$$\hat{\mathbf{a}}_{r'} = \sin\theta'\cos\phi'\hat{\mathbf{a}}_x + \sin\theta'\sin\phi'\hat{\mathbf{a}}_y + \cos\theta'\hat{\mathbf{a}}_z$$

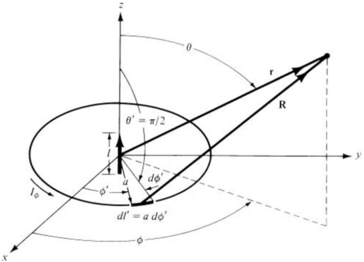
- The angle ψ between the two directions,

$$\cos\psi = \hat{\mathbf{a}}_r \cdot \hat{\mathbf{a}}_{r'}$$

$$\boxed{\cos\psi = \sin\theta\sin\theta'\cos\left(\phi - \phi'\right) + \cos\theta\cos\theta'}$$

Circumference $C < \frac{\lambda}{10}$

$$\mathbf{A}(r,\theta,\phi) = \frac{\mu}{4\pi}\int_C \mathbf{I}_e \frac{e^{-jkR}}{R} dl'$$



The source current is along $\hat{\mathbf{a}}_{\phi'}$: $\mathbf{I}_e = I_0\hat{\mathbf{a}}_{\phi'}$. Transforming to Cartesian coordinates,

$$\begin{aligned} I_x &= -I_0 \sin\phi', \\ I_y &= I_0 \cos\phi' \end{aligned}$$

Transforming to the spherical coordinates at the observation point,

$$\begin{aligned} I_r &= I_x \cos\phi \sin\theta + I_y \sin\phi \sin\theta \\ I_\theta &= I_x \cos\phi \cos\theta + I_y \sin\phi \cos\theta \\ I_\phi &= -I_x \sin\phi + I_y \cos\phi \end{aligned}$$

Small Circular Loop

$$I_x = -I_0 \sin\phi',$$
$$I_y = I_0 \cos\phi'$$

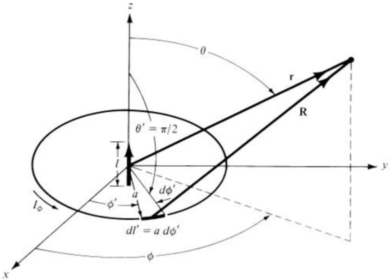
Transforming to the spherical coordinates at the observation point,

$$I_r = I_x \cos\phi \sin\theta + I_y \sin\phi \sin\theta$$
$$I_\theta = I_x \cos\phi \cos\theta + I_y \sin\phi \cos\theta$$
$$I_\phi = -I_x \sin\phi + I_y \cos\phi$$

$$I_r = I_0 \sin\theta \sin\left(\phi - \phi'\right)$$
$$I_\theta = I_0 \cos\theta \sin\left(\phi - \phi'\right)$$
$$I_\phi = I_0 \cos\left(\phi - \phi'\right)$$

$$\mathbf{I}_e = \hat{\mathbf{a}}_r I_0 \sin\theta \sin\left(\phi - \phi'\right) + \hat{\mathbf{a}}_\theta I_0 \cos\theta \sin\left(\phi - \phi'\right) + \hat{\mathbf{a}}_\phi I_0 \cos\left(\phi - \phi'\right)$$

Small Circular Loop



$$R = \sqrt{r^2 + a^2 - 2ar\cos\psi}, \qquad \cos\psi = \sin\theta\cos\left(\phi - \phi'\right)$$

$$A_r = \frac{\mu I_0 a}{4\pi}\sin\theta\int_0^{2\pi}\sin\left(\phi-\phi'\right)\frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos\left(\phi-\phi'\right)}}d\phi' = 0$$

$$A_\theta = \frac{\mu I_0 a}{4\pi}\cos\theta\int_0^{2\pi}\sin\left(\phi-\phi'\right)\frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos\left(\phi-\phi'\right)}}d\phi' = 0$$

$$A_\phi = \frac{\mu I_0 a}{4\pi}\int_0^{2\pi}\cos\left(\phi-\phi'\right)\frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos\left(\phi-\phi'\right)}}d\phi'$$

Small Circular Loop

$$\mathbf{A} = A_\phi \hat{\mathbf{a}}_\phi, \quad A_\phi = \frac{\mu I_0 a}{4\pi} \int_0^{2\pi} \cos(\phi - \phi') \frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}} d\phi'$$

$$f = \frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}$$

$$f(a) \backsimeq f(0) + f'(0)\, a$$

$$f(0) = \frac{e^{-jkr}}{r}, \qquad f'(0) = \frac{e^{-jkr}}{r^2}(jkr+1)\sin\theta\cos(\phi - \phi')$$

$$f = \frac{e^{-jkr}}{r}\left[1 + a\left(\frac{1}{r} + jk\right)\sin\theta\cos(\phi-\phi')\right]$$

$$A_\phi = \frac{\mu I_0 a}{4\pi}\frac{e^{-jkr}}{r}\int_0^{2\pi}\cos(\phi-\phi')\left[1 + a\left(\frac{1}{r}+jk\right)\sin\theta\cos(\phi-\phi')\right]d\phi'$$

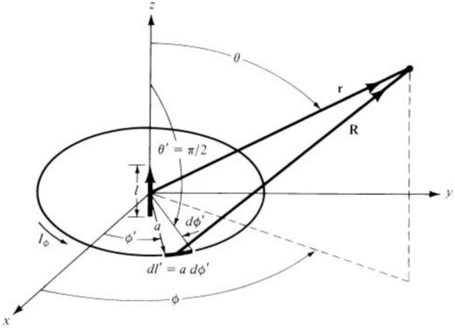
$$A_\phi = \frac{a^2 jk\mu I_0}{4}\frac{e^{-jkr}}{r}\left(\frac{1}{jkr}+1\right)\sin\theta$$

$$\mathbf{A} = A_\phi \hat{\mathbf{a}}_\phi,$$

$$A_\phi = \frac{a^2 jk\mu I_0}{4} \frac{e^{-jkr}}{r} \left(\frac{1}{jkr} + 1 \right) \sin\theta$$

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A},$$

$$\mathbf{E} = \frac{1}{j\omega\varepsilon} \nabla \times \mathbf{H} = \frac{\eta}{jk} \nabla \times \mathbf{H}$$



$$\mathbf{H} = -\frac{(ka)^2 I_0 e^{-jkr}}{4r} \left[\left(\frac{2}{jkr} - \frac{2}{(kr)^2} \right) \cos\theta \hat{\mathbf{a}}_r + \left(-\frac{1}{(kr)^2} + \frac{1}{jkr} + 1 \right) \sin\theta \hat{\mathbf{a}}_\theta \right]$$

$$\mathbf{E} = \eta \frac{(ka)^2 I_0 e^{-jkr}}{4r} \left(\frac{1}{jkr} + 1 \right) \sin\theta \hat{\mathbf{a}}_\phi$$

Small Circular Loop

- Far Fields,

$$\mathbf{H} = -\frac{(ka)^2 I_0 e^{-jkr}}{4r} \sin\theta \hat{\mathbf{a}}_\theta, \qquad \mathbf{E} = \eta \frac{(ka)^2 I_0 e^{-jkr}}{4r} \sin\theta \hat{\mathbf{a}}_\phi$$

- Radiated Power and Radiation Resistance

$$U = \eta \frac{(ka)^4 |I_0|^2}{32} \sin^2\theta, \qquad P_{\text{rad}} = \int U d\Omega = \eta \frac{\pi (ka)^4 |I_0|^2}{12}$$

$$\boxed{R_r = \eta \left(\frac{\pi}{6}\right) (ka)^4}$$

For N-turns loop,

$$R_r = R_r = \eta \left(\frac{\pi}{6}\right) (ka)^4 N^2$$

- Loss resistance R_L,

$$R_L = N \frac{a}{b} R_s,$$

where b is the wire radius, and R_s is the conductor surface impedance.

Increasing the number of turns, increases the antenna radiation efficiency.