Topic 2
Radiation Field Integrals

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Radiation Integrals

Outline

1. Maxwell's Equations
2. Vector and Scalar Potentials
3. Far Field Apprximation

Maxwell's Equations

Instantaneous Fields

$$\nabla \times \mathscr{E}(\mathbf{r},t) = -\frac{\partial \mathscr{B}(\mathbf{r},t)}{\partial t}$$

$$\nabla \times \mathscr{H}(\mathbf{r},t) = \frac{\partial \mathscr{D}(\mathbf{r},t)}{\partial t} + \mathscr{J}(\mathbf{r},t)$$

$$\nabla \cdot \mathscr{D} = \rho_t(\mathbf{r},t)$$

$$\nabla \cdot \mathscr{B} = 0$$

Constitutive relations for free space,

$$\mathscr{D} = \varepsilon_0 \mathscr{E}$$

$$\mathscr{B} = \mu_0 \mathscr{H}$$

Phasor Fields

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = \mathbf{0}$$

$$\nabla \cdot \mathbf{D} = \rho$$

- Constitutive relations

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

Outline

1. Maxwell's Equations

2. Vector and Scalar Potentials

3. Far Field Apprximation

Vector and Scalar Potentials

$$\nabla \cdot \mathbf{B} = 0, \quad \Longrightarrow \quad \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \quad \Longrightarrow \quad \nabla \times (\mathbf{E} + j\omega \mathbf{A}) = \mathbf{0}$$

$$\mathbf{E} + j\omega \mathbf{A} = -\nabla \Phi_e \quad \Longrightarrow \quad \boxed{\mathbf{E} = -\nabla \Phi_e - j\omega \mathbf{A}}$$

- Substituting in Maxwell's equation

$$\nabla \times \mathbf{H} = j\omega\varepsilon \mathbf{E} + \mathbf{J}$$

$$\nabla \times \nabla \times \mathbf{A} = -j\omega\mu\varepsilon (\nabla \Phi_e + j\omega \mathbf{A}) + \mu \mathbf{J}$$

$$\nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = -j\omega\mu\varepsilon \nabla \Phi_e + k^2 \mathbf{A} + \mu \mathbf{J}$$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = \nabla (j\omega\mu\varepsilon \Phi_e + \nabla \cdot \mathbf{A}) - \mu \mathbf{J}$$

  - One gauge choice is called Lorentz gauge where,

$$j\omega\mu\varepsilon \Phi_e + \nabla \cdot \mathbf{A} = 0$$

$$\boxed{\mathbf{E} = -j\omega \left[\mathbf{A} + \frac{1}{k^2} \nabla (\nabla \cdot \mathbf{A}) \right]}$$

Integral Solution of the Fields

- Lorentz gauge

$$j\omega\mu\varepsilon\Phi_e + \nabla \cdot \mathbf{A} = 0$$

- The differential equation for **A**,

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

- Similarly Φ_e satisfies,

$$\nabla^2 \Phi_e + k^2 \Phi_e = -\rho/\varepsilon$$

- Integral Solution for **A**, and Φ_e,

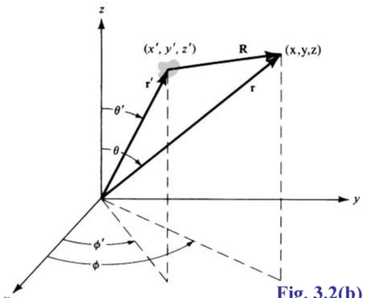

Fig. 3.2(b)

$$\mathbf{A}(r,\theta,\phi) = \frac{\mu}{4\pi} \iiint_V \mathbf{J} \frac{e^{-jkR}}{R} dv'$$

$$\Phi_e(r,\theta,\phi) = \frac{1}{4\pi\varepsilon_0} \iiint_V \rho \frac{e^{-jkR}}{R} dv'$$

Outline

$$\mathbf{A}(r,\theta,\phi) = \frac{\mu}{4\pi}\iiint_V \mathbf{J}\frac{e^{-jkR}}{R}dv'$$

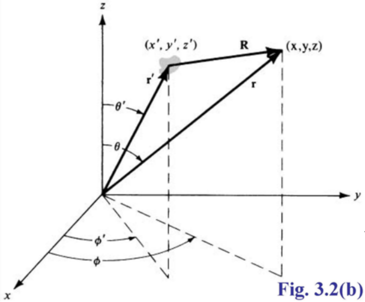

Fig. 3.2(b)

Far field approximations $r \gg r'$:

$$R = \sqrt{(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')} = \sqrt{r^2 + r'^2 - 2\mathbf{r}'\cdot\mathbf{r}}$$

$$R \backsimeq r - \mathbf{r}'\cdot\hat{\mathbf{a}}_r, \quad kR \backsimeq kr - \mathbf{k}\cdot\mathbf{r}', \quad \frac{1}{R} \backsimeq \frac{1}{r},$$

where $\mathbf{k} = k\hat{\mathbf{a}}_r$.

$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint_V \mathbf{J}e^{j\mathbf{k}\cdot\mathbf{r}'}dv'$$

Far Field Apprximation for Magnetic vector Potential

$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{e^{-jkr}}{r}\mathbf{A}^0(\theta,\phi), \qquad \text{where } \mathbf{A}^0(\theta,\phi) = \frac{\mu}{4\pi}\iiint_V \mathbf{J}e^{j\mathbf{k}\cdot\mathbf{r}'}dv'$$

Integral Solution of the Fields

Far Field approximation

Radiation Magnetic Vector Potential A

$$\mathbf{A}(r,\theta,\phi) = \frac{e^{-jkr}}{r}\mathbf{A}^0(\theta,\phi), \qquad \text{where } \mathbf{A}^0(\theta,\phi) = \frac{\mu_0}{4\pi}\iiint_V \mathbf{J}e^{j\mathbf{k}\cdot\mathbf{r}'}dv'$$

$$\nabla\times\mathbf{A} = \nabla\left(\frac{e^{-jkr}}{r}\right)\times\mathbf{A}^0(\theta,\phi) + \frac{e^{-jkr}}{r}\nabla\times\mathbf{A}^0(\theta,\phi)$$

$$\begin{aligned}\nabla\times\mathbf{A}^0 \quad = \quad & \frac{\hat{\mathbf{a}}_r}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(A^0_\phi\sin\theta\right) - \frac{\partial A^0_\theta}{\partial\phi}\right] + \frac{\hat{\mathbf{a}}_\theta}{r}\left[\frac{1}{\sin\theta}\frac{\partial A^0_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA^0_\phi\right)\right] \\ & + \frac{\hat{\mathbf{a}}_\phi}{r}\left[\frac{\partial}{\partial r}\left(rA^0_\theta\right) - \frac{\partial A^0_r}{\partial\theta}\right] = \mathcal{O}\left(\frac{1}{r}\right)\end{aligned}$$

$$\nabla\times\mathbf{A} = \nabla\left(\frac{e^{-jkr}}{r}\right)\times\mathbf{A}^0(\theta,\phi) + \cancel{\frac{e^{-jkr}}{r}\nabla\times\mathbf{A}^0(\theta,\phi)} \rightarrow \mathcal{O}\left(\frac{1}{r^2}\right)$$

Far Field Approximation of the Fields

$$\nabla \times \mathbf{A} = \nabla \left(\frac{e^{-jkr}}{r} \right) \times \mathbf{A}^0(\theta, \phi) + \cancelto{\mathcal{O}\left(\frac{1}{r^2}\right)}{\frac{e^{-jkr}}{r} \nabla \times \mathbf{A}^0(\theta, \phi)}$$

$$\nabla \times \mathbf{A} = -jk \frac{e^{-jkr}}{r} \hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta, \phi) - \cancelto{\mathcal{O}\left(\frac{1}{r^2}\right)}{\frac{e^{-jkr}}{r^2} \hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta, \phi)}$$

$$+ \cancelto{\mathcal{O}\left(\frac{1}{r^2}\right)}{\frac{e^{-jkr}}{r} \nabla \times \mathbf{A}^0(\theta, \phi)}$$

Far Field Approximation of $\nabla \times \mathbf{A}$

$$\boxed{\nabla \times \mathbf{A} = -jk \hat{\mathbf{a}}_r \times \mathbf{A}, \qquad \text{where } \mathbf{A}(r, \theta, \phi) = \frac{e^{-jkr}}{r} \mathbf{A}^0(\theta, \phi)}$$

Far Field Approximation of the Fields

Far Field Approximation of $\nabla \times \mathbf{A}$

$$\boxed{\nabla \times \mathbf{A} = -jk\hat{\mathbf{a}}_r \times \mathbf{A}, \qquad \text{where } \mathbf{A}(r,\theta,\phi) = \frac{e^{-jkr}}{r}\mathbf{A}^0(\theta,\phi)}$$

- The magnetic field $\mathbf{H}$ is approximated in the far field as,

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \backsimeq \frac{-jk}{\mu}\hat{\mathbf{a}}_r \times \mathbf{A} = -j\omega\sqrt{\frac{\varepsilon}{\mu}}\frac{e^{-jkr}}{r}\hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta,\phi)$$

$$\mathbf{H} = \frac{e^{-jkr}}{r}\mathbf{H}^0(\theta,\phi), \qquad \text{where } \mathbf{H}^0(\theta,\phi) = -j\omega\sqrt{\frac{\varepsilon}{\mu}}\hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta,\phi)$$

- The electric field $\mathbf{E}$ far away from the source where $\mathbf{J} = 0$, satisfies,

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

- In the far field region the electric field can be approximated as,

$$\mathbf{E} = \frac{1}{j\omega\varepsilon}\nabla \times \mathbf{H} \backsimeq \frac{-jk}{j\omega\varepsilon}\hat{\mathbf{a}}_r \times \mathbf{H} = -\sqrt{\frac{\mu}{\varepsilon}}\hat{\mathbf{a}}_r \times \mathbf{H}$$

Far Field Approximation of the Fields

- The magnetic field $\mathbf{H}$ is approximated in the far field as,

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \backsimeq \frac{-jk}{\mu}\hat{\mathbf{a}}_r \times \mathbf{A} = \frac{-j\omega}{\eta}\hat{\mathbf{a}}_r \times \mathbf{A} = \frac{-j\omega}{\eta}\frac{e^{-jkr}}{r}\hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta,\phi)$$

$$\mathbf{H} = \frac{e^{-jkr}}{r}\mathbf{H}^0(\theta,\phi), \qquad \text{where } \mathbf{H}^0(\theta,\phi) = -j\omega\sqrt{\frac{\varepsilon}{\mu}}\hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta,\phi)$$

- In the far field region the electric field $\mathbf{E}$ can be approximated as,

$$\mathbf{E} = -\eta\hat{\mathbf{a}}_r \times \mathbf{H} = j\omega\hat{\mathbf{a}}_r \times (\hat{\mathbf{a}}_r \times \mathbf{A})$$

- Using the vector identity, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$,

$$\mathbf{E} = j\omega\hat{\mathbf{a}}_r \times (\hat{\mathbf{a}}_r \times \mathbf{A}) = j\omega[A_r\hat{\mathbf{a}}_r - \mathbf{A}] = -j\omega\left(A_\theta\hat{\mathbf{a}}_\theta + A_\phi\hat{\mathbf{a}}_\phi\right)$$

Electric Field E

$$\mathbf{E} = -j\omega \left(A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi \right)$$

$$\mathbf{E} = \frac{e^{-jkr}}{r}\mathbf{E}^0\left(\theta,\phi\right), \text{ where } \mathbf{E}^0 = E^0_\theta \hat{\mathbf{a}}_\theta + E^0_\phi \hat{\mathbf{a}}_\phi$$

Magnetuc Field H

$$\mathbf{H} = \frac{1}{\eta}\hat{\mathbf{a}}_r \times \mathbf{E}$$

$$\mathbf{H} = \frac{e^{-jkr}}{r}\mathbf{H}^0\left(\theta,\phi\right), \text{ where } \mathbf{H}^0\left(\theta,\phi\right) = \frac{1}{\eta}\hat{\mathbf{a}}_r \times \mathbf{E}\left(\theta,\phi\right)$$

In the far field the radiation fields constitute spherical TEM wave.

Integral Solution of the Fields

Far Field approximation

$$\mathbf{H} = \frac{1}{\mu}\nabla \times \mathbf{A} \backsimeq -j\omega\sqrt{\frac{\varepsilon}{\mu}}\frac{e^{-jkr}}{r}\hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta,\phi)$$

Radiation Electric and Magnetic Fields

$$\mathbf{H} \backsimeq \frac{1}{\eta}\hat{\mathbf{a}}_r \times \mathbf{E}, \qquad \mathbf{E} \backsimeq -j\omega\left[A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi\right]$$

$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iiint_V \mathbf{J}(r',\theta',\phi')\, e^{j\mathbf{k}\cdot\mathbf{r}'}dv'$$

- Surface current

$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\iint_S \mathbf{J}_s(r',\theta',\phi')\, e^{j\mathbf{k}\cdot\mathbf{r}'}ds'$$

- Linear current

$$\mathbf{A}(r,\theta,\phi) \backsimeq \frac{\mu}{4\pi}\frac{e^{-jkr}}{r}\int_C \mathbf{I}_e(r',\theta',\phi')\, e^{j\mathbf{k}\cdot\mathbf{r}'}dl'$$

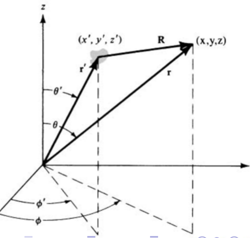

Fig. 3.2(