Small Signal Space Charge Modes in a Guiding Structure

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After some mathematical manipulations, we were able to reach to the following PDE that determies the axial electric field E_z.

$$\nabla_{\perp}^2 E_z + T^2 E_z = 0$$

where,

$$T^2 = \begin{cases} T_n^2 = \left(\gamma^2 - k^2\right)\left[\frac{\beta_p^2}{(\beta_e - \gamma)^2} - 1\right] & \text{inside the beam } r < b \\ -\tau_n^2 = -\left(\gamma^2 - k^2\right) & \text{outside the beam } b < r < a \end{cases} \tag{1}$$

$$E_z = \begin{cases} BJ_0\left(T_n r\right), & \text{inside the beam } r < b \\ CI_0\left(\tau_n r\right) + DK_0\left(\tau_n r\right) & \text{outside the beam } b < r < a \end{cases}$$

To satisfy the boundary conditions at $r = a$, $\implies$ $E_z\left(r = a\right) = 0$,

$$E_z = \begin{cases} BJ_0\left(T_n r\right), & \text{inside the beam } r < b \\ C'\left[I_0\left(\tau_n r\right)K_0\left(\tau_n a\right) - K_0\left(\tau_n r\right)I_0\left(\tau_n a\right)\right] & \text{outside the beam } b < r < a \end{cases}$$

The other components of the fields can be obtained using Maxwell's equations,

$$\nabla \times \mathbf{E} = -j\omega\mu_0 H_\phi \hat{\mathbf{a}}_\phi \quad \implies \quad -j\gamma E_r - \frac{\partial E_z}{\partial r} = -j\omega\mu_0 H_\phi$$

$$\nabla \times \mathbf{H} = j\omega\epsilon_0 \mathbf{E} + J_z \hat{\mathbf{a}}_z, \quad \implies \quad j\gamma H_\phi = j\omega\epsilon_0 E_r$$

From these two equations,

$$H_\phi = \frac{-j\omega\epsilon_0}{k^2 - \gamma^2}\frac{\partial E_z}{\partial r}$$

$$H_\phi = \frac{-j\omega\epsilon_0}{k^2 - \gamma^2}\begin{cases} BT_n J_0'\left(T_n r\right) & \text{inside the beam } r < b \\ C\tau_n\left[I_0'\left(\tau_n r\right)K_0\left(\tau_n a\right) - K_0'\left(\tau_n r\right)I_0\left(\tau_n a\right)\right] & \text{outside the beam } b < r < a \end{cases}$$

Apply the continuity of E_z and H_ϕ at $r = b$,

$$\begin{aligned} BJ_0\left(T_n b\right) &= C'\left[I_0\left(\tau_n b\right)K_0\left(\tau_n a\right) - K_0\left(\tau_n b\right)I_0\left(\tau_n a\right)\right] \\ BT_n J_0'\left(T_n b\right) &= C'\tau_n\left[I_0'\left(\tau_n b\right)K_0\left(\tau_n a\right) - K_0'\left(\tau_n b\right)I_0\left(\tau_n a\right)\right] \end{aligned}$$

Dividing these two equations,

$$T_n\frac{J_0'\left(T_n b\right)}{J_0\left(T_n b\right)} = \tau_n\frac{I_0'\left(\tau_n b\right)K_0\left(\tau_n a\right) - K_0'\left(\tau_n b\right)I_0\left(\tau_n a\right)}{I_0\left(\tau_n b\right)K_0\left(\tau_n a\right) - K_0\left(\tau_n b\right)I_0\left(\tau_n a\right)}$$

Using these identities,

$$J_0'(x) = -J_1(x), \quad I_0'(x) = I_1(x), \qquad K_0'(x) = -K_1(x)$$

$$\boxed{T_n \frac{J_1(T_n b)}{J_0(T_n b)} = \tau_n \frac{I_1(\tau_n b) K_0(\tau_n a) + K_1(\tau_n b) I_0(\tau_n a)}{K_0(\tau_n b) I_0(\tau_n a) - I_0(\tau_n b) K_0(\tau_n a)}} \tag{2}$$

Assumptions

We will assume the phase velocities almost equal to the velocity of the beam making $\gamma \approx \beta_e$, and hence $\gamma \gg k$. We can then use β_e instead of γ in Eq. (1) (except in the denominator term), with the result,

$$\tau_n \approx \beta_e \tag{3}$$

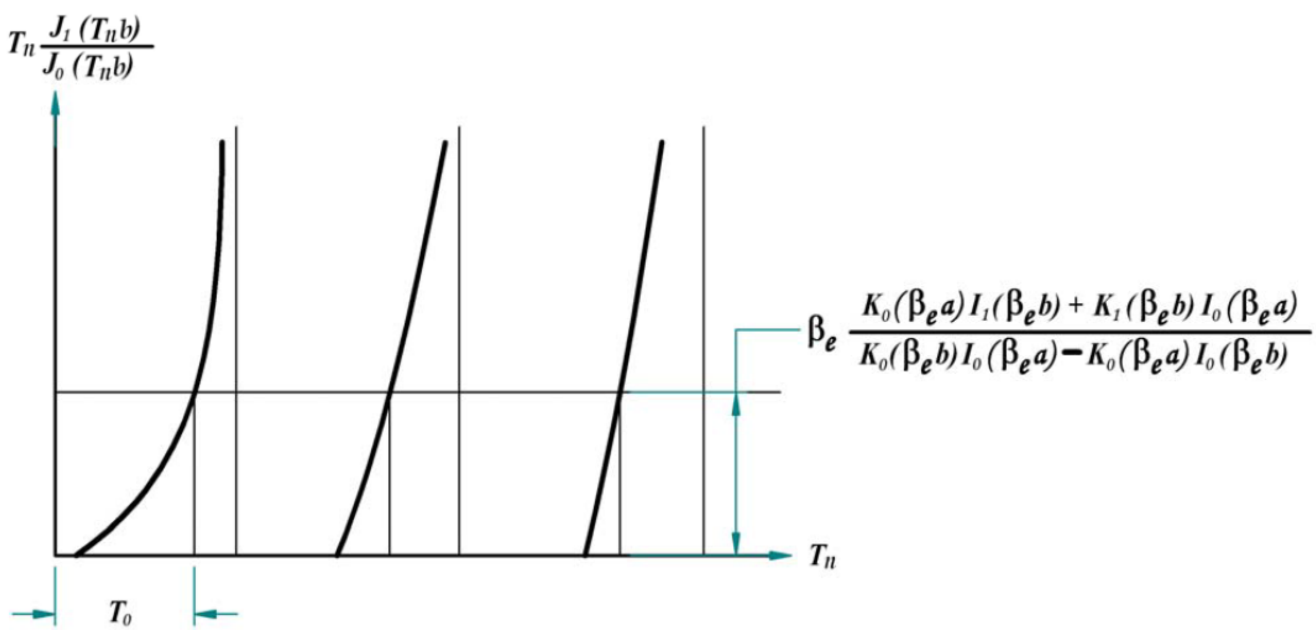
$$T_n \approx \beta_e \left[\frac{\beta_p^2}{(\beta_e - \gamma)^2} - 1 \right], \quad \Longrightarrow \quad \gamma = \beta_e \pm \beta_q = \beta_e \pm \frac{\beta_p}{\sqrt{1 + \frac{T_n^2}{\beta_e^2}}}$$

The plasma reduction factor R is defined as,

$$R = \frac{\beta_q}{\beta_p} = \frac{1}{\sqrt{1 + \frac{T_n^2}{\beta_e^2}}},$$

where T_n is obtained by solving equation (2), after using the approximation Eq. (3),

$$\boxed{T_n \frac{J_1(T_n b)}{J_0(T_n b)} = \beta_e \frac{I_1(\beta_e b) K_0(\beta_e a) + K_1(\beta_e b) I_0(\beta_e a)}{K_0(\beta_e b) I_0(\beta_e a) - I_0(\beta_e b) K_0(\beta_e a)}} \tag{4}$$



The dispersion relation can be written in normalized form as,

$$\boxed{(T_n b)\,\frac{J_1\left(T_n b\right)}{J_0\left(T_n b\right)} = (\beta_e b)\,\frac{I_1\left(\beta_e b\right) K_0\left(\beta_e b\frac{a}{b}\right) + K_1\left(\beta_e b\right) I_0\left(\beta_e b\frac{a}{b}\right)}{K_0\left(\beta_e b\right) I_0\left(\beta_e b\frac{a}{b}\right) - I_0\left(\beta_e b\right) K_0\left(\beta_e b\frac{a}{b}\right)}} \tag{5}$$

This dispersion equation is solved for the different modes $T_n b$ for the two input independent variables $\beta_e b$ and $\frac{a}{b}$, and the plasma reduction factor is given as,

$$R = \frac{\beta_q}{\beta_p} = \frac{1}{\sqrt{1 + \left(\frac{T_n b}{\beta_e b}\right)^2}},$$