DFT

Course Information

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TAs: Ahmed AbdelKarim & Hazem Soliman

Lectures: Sundays and Wednesdays

Grading: Midterm 22%, Projects: 8%, Final 70%

Lec.	Topic	Source
1	Introduction and Fourier Transforms	Chapter 5, Proakis DSP book
2	**DFT:** • Sampling in the frequency domain • Time-Domain aliasing	Chapter 5, Proakis DSP book
3	**DFT:** • Properties • Circular convolution and linear convolution using circular convolution	Chapter 5, Proakis DSP book
4	**DFT:** • Frequency resolution and windowing **Wireline Channel** • Properties • Interference sources	Chapter 5, Proakis DSP book Lecture notes
5	**Fading:** • Origin of fading • Doppler frequency • Classification of fading channels	Chapter 4Rappaport
6	**Fading:** • Fast and slow channels • Flat and frequency selective channels	Chapter 4 Rappaport
7	**Fading:** • Delay spread and coherence bandwidth • Doppler spread and coherence bandwidth	Chapter 4 Rappaport

8	**Multichannel Modulation (MCM):** • Advantages and how MCM combats ISI • Block diagram of MCM transceiver • Basis functions	Haykin, section 6.12
9	**MCM:** • The water-filling algorithm	Haykin, section 6.12
10	**Discrete Multi-tone DMT:** • Using DFT symmetry properties to generate real baseband MCM signal • DSL basics	Haykin, section 6.12, and Cioffi's tutorial
11	**OFDM:** • Properties of the wireless channel and introduction to multipath fading, and the delay spread • Advantages and disadvantages of OFDM systems in wireless channels • Guard time and cyclic extension	Prasad's OFDM book
12	**OFDM:** • Block diagram of a "digital" OFDM transceiver • Choice of OFDM parameters	Prasad's OFDM book

Wireless Communications

History of Wireless

The Birth of Radio

- 1897 — “The Birth of Radio” - Marconi awarded patent for wireless telegraph
- 1897 — First “Marconi station” established on Needles island to communicate with English coast
- 1898 — Marconi awarded English patent no. 7777 for tuned communication
- 1898 — Wireless telegraphic connection between England and France established

Transoceanic Communication

- 1901 — Marconi successfully transmits radio signal across Atlantic Ocean from (first wireless communication across the ocean) Cornwall to Newfoundland
- 1902 — First bidirectional communication across Atlantic
- 1909 — Marconi awarded Nobel prize for physics

http://wireless.ece.ufl.edu/jshea/wireless_history.html

History of Wireless (2)

Voice over Radio

- 1914 — First voice over radio transmission
- 1920s — Mobile receivers installed in police cars in Detroit
- 1930s — Mobile transmitters developed; radio equipment occupied most of police car trunk
- 1935 — Frequency modulation (FM) demonstrated by Armstrong
- 1940s — Majority of police systems converted to FM

Birth of Mobile Telephony

- 1946 — First interconnection of mobile users to public switched telephone network (PSTN)
- 1949 — FCC recognizes mobile radio as new class of service
- 1940s — Number of mobile users > 50K
- 1950s — Number of mobile users > 500K
- 1960s — Number of mobile users > 1.4M
- 1960s — Improved Mobile Telephone Service (IMTS) introduced; supports full-duplex, auto dial, auto trunking
- 1976 — Bell Mobile Phone has 543 pay customers using 12 channels in the New York City area; waiting list is 3700 people; service is poor due to blocking

History of Wireless (3)

Cellular Mobile Telephony

- 1979 — NTT/Japan deploys first cellular communication system
- 1983 — Advanced Mobile Phone System (AMPS) deployed in US in 900 MHz band: supports 666 duplex channels
- 1989 — Groupe Spècial Mobile defines European digital cellular standard, GSM
- 1991 — US Digital Cellular phone system introduced
- 1993 — IS-95 code-division multiple-access (CDMA) spread- spectrum digital cellular system deployed in US
- 1994 — GSM system deployed in US, relabeled ``Global System for Mobile Communications''

Wireless Local Area Networks

- 1990 — Formation of IEEE 802.11 Working Group to define standards for Wireless Local Area Networks (WLANs)
- 1997-2003 — Releases of IEEE 802.11 WLAN protocol, supporting 1-54 Mbit/s data rates in the 2.4/5 GHz ISM bands based on Orthogonal Frequency Division Multiplexing (OFDM)
- 2009 — Release of IEEE 802.11n WLAN protocol, supporting up to 150 Mbit/s data rates in both the 2.4 GHz and 5 GHz ISM bands.

History of Wireless (4)

Wireless Metropolitan Area Networks

- 1999 — Formation of IEEE 802.16 Working Group to define standards for Wireless Metropolitan Area Networks (WLANs)
- 2004 — release of 802.16d (fixed WiMAX standard) (OFDM)
- 2005 —release of 802.16e (Mobile WiMAX standard)
- 2009 —Cairo University hosted the WIMAX standard meeting to discuss development of WiMAX release 2
- 2012 — WiMAX release 2 commercially available

History of Wireless (5)

3G networks and beyond

- 2001 — UMTS deployment based on WCDMA and CDMA2000
- 2007 — HSPA often referred to as 3.5G supporting 14Mbps on the downlink
- 2008 — HSPA+ often referred to as 3.75G supporting 42Mbps on the downlink
- 2010 — Number of cellular phones surpassed 4 billion worldwide and 65 million in Egypt.
- 2009 — first LTE (long term evolution) system deployment is Sweden supporting 100Mbps on the downlink. LTE is based on OFDM

Fourier transforms

	Freq domain	Time domain
Fourier transform		
Fourier series		
Discrete time Fourier transform		

Fourier transforms

	Freq domain	Time domain
Fourier transform	Cont/aperiodic	Aperiodic/cont
Fourier series		
Discrete time Fourier transform		

Fourier transforms

	Freq domain	Time domain
Fourier transform	Cont/aperiodic	Aperiodic/cont
Fourier series	Disc/aperiodic	Periodic/cont
Discrete time Fourier transform		

Fourier transforms

	Freq domain	Time domain
Fourier transform	Cont/aperiodic	Aperiodic/cont
Fourier series	Disc/aperiodic	Periodic/cont
Discrete time Fourier transform	cont/periodic	Aperiodic/disc

Fourier Transform

$$x(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$x(t) = \int_{-\infty}^{\infty} x(f)e^{+j2\pi ft}df$$

Fourier Series

$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi f_n t}dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_n t}$$

$$f_n = n/T$$

DTFT

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{+j\omega n} d\omega$$

Discrete Fourier Transform (DFT)

Motivation

- We need a transform that is discrete in both domains, to be able to manipulate signals on processors.
- For example, given a discrete time signal, we need a DISCRETE frequency domain representation, unlike the DTFT which is continuous in the frequency domain.

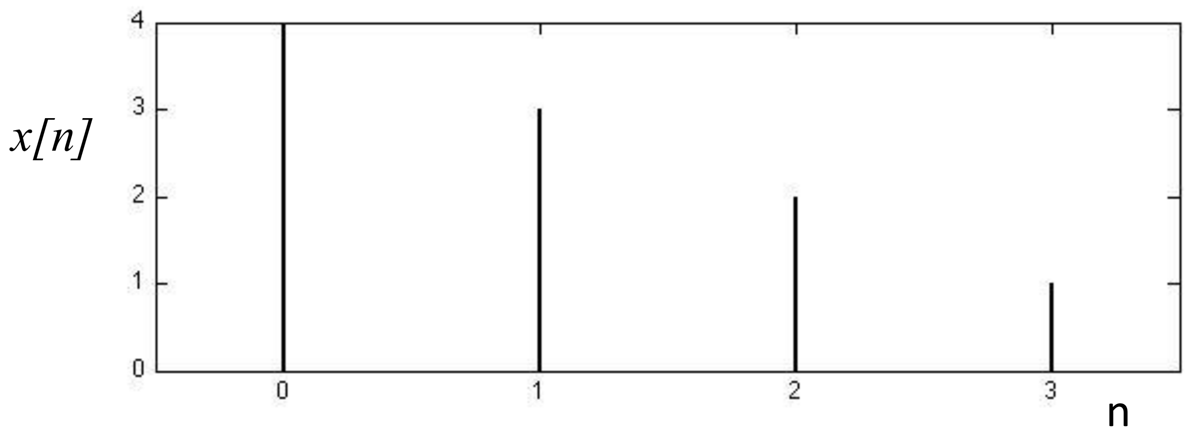
Approach

- Analogous to sampling in the time domain, we will consider sampling the DTFT. Remember that sampling in the TD causes repetition of the spectrum in the FD.

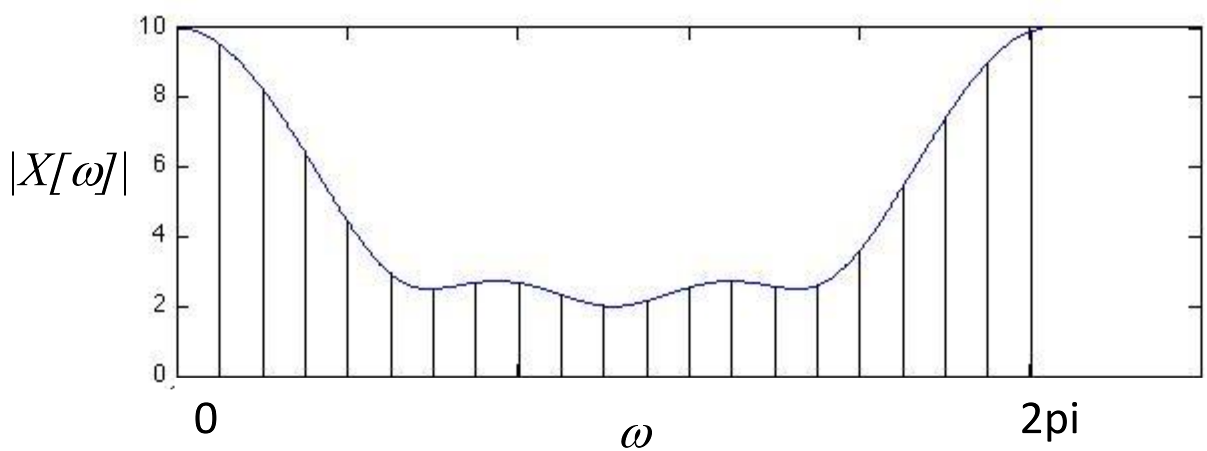
$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Take $\omega_k = \frac{2\pi}{N} k$

Sampling in the Freq Domain



Time domain



Frequency domain
(magnitude)

Sampling in the Freq Domain (2)

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\frac{2\pi k n}{N}}$$

$$= \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi\frac{kn}{N}} + \sum_{n=0}^{N-1} x(n)\,e^{-j2\pi\frac{kn}{N}} + \cdots$$

Sampling in the Freq Domain (3)

$$X(\frac{2\pi k}{N}) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN) e^{-j2\pi kn/N}$$

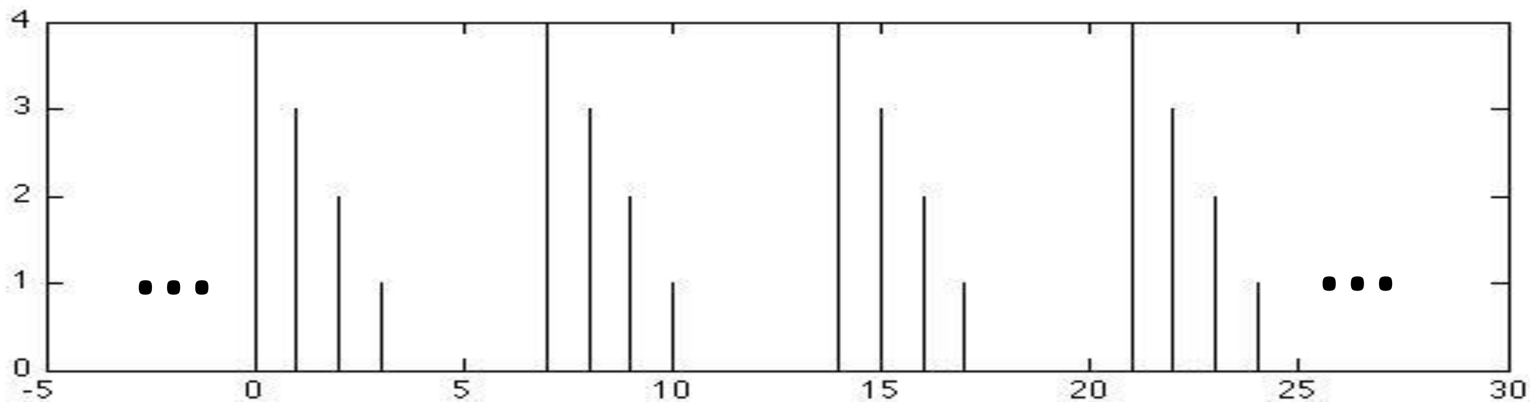
We define a new signal $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$

$$X(\frac{2\pi k}{N}) = \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N}$$

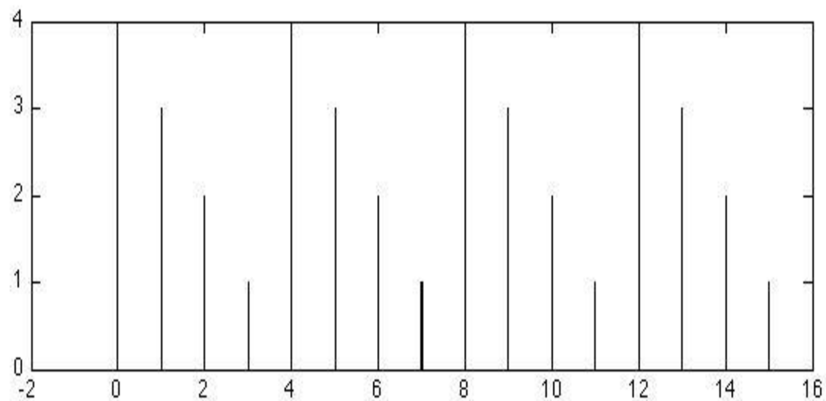
$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(\frac{2\pi k}{N}) e^{j2\pi kn/N}$$

Fourier Transform pair
The DFT

$$x_p(n)$$



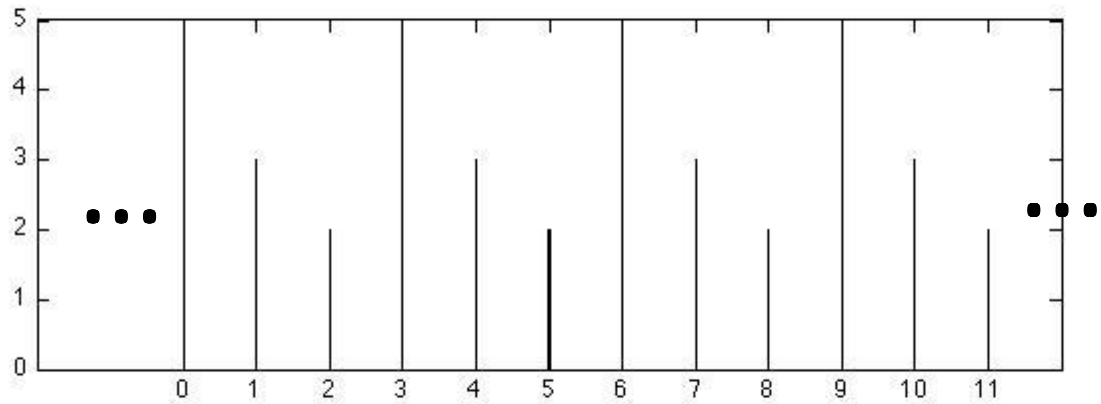
N>L



N=L

Time Domain Aliasing

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

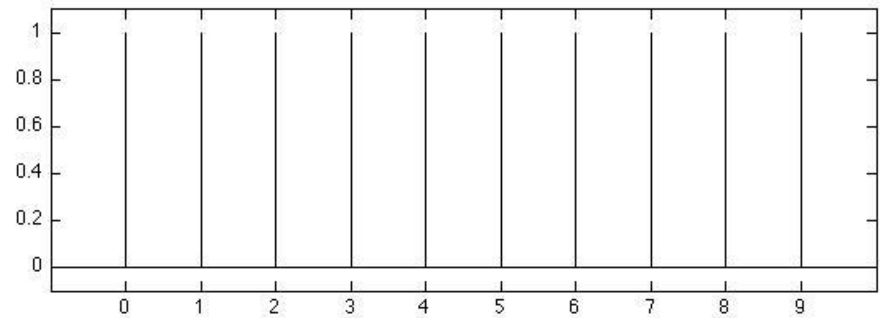


N<L

Example

Find the DFT, X(k), of

$$x(n) = \begin{cases} 1 & 0 \le \text{n} \le \text{L-1} \\ 0 & \text{otherwise} \end{cases}$$

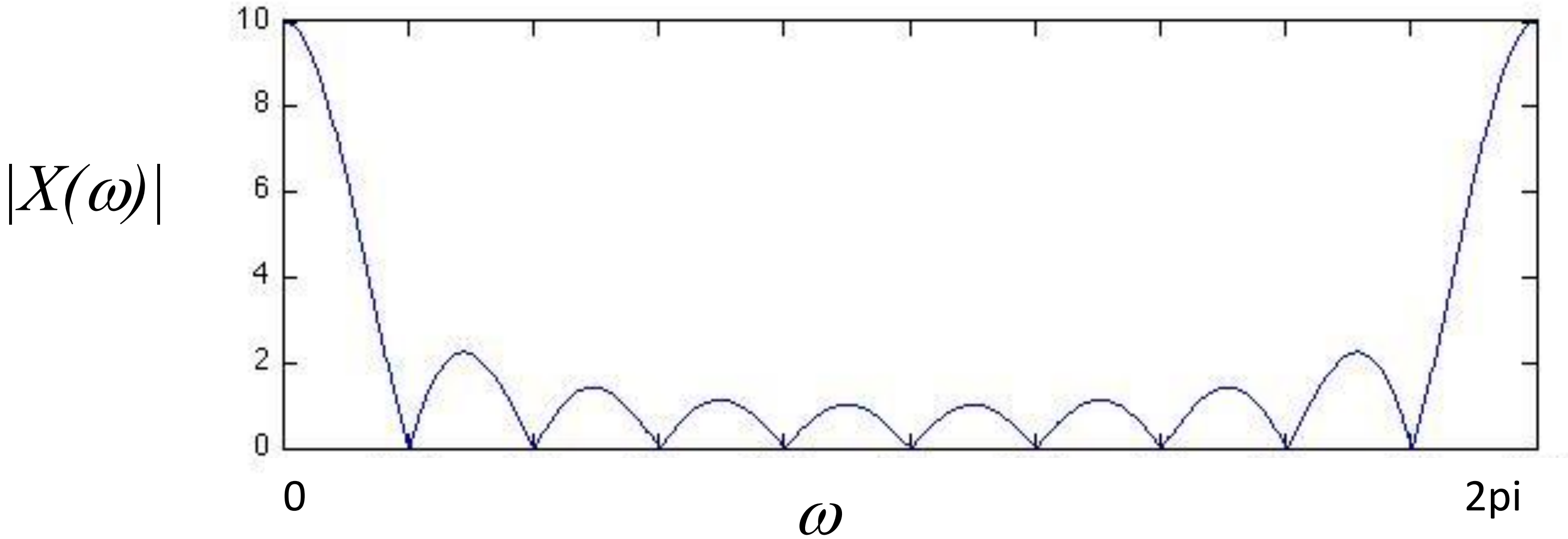


n

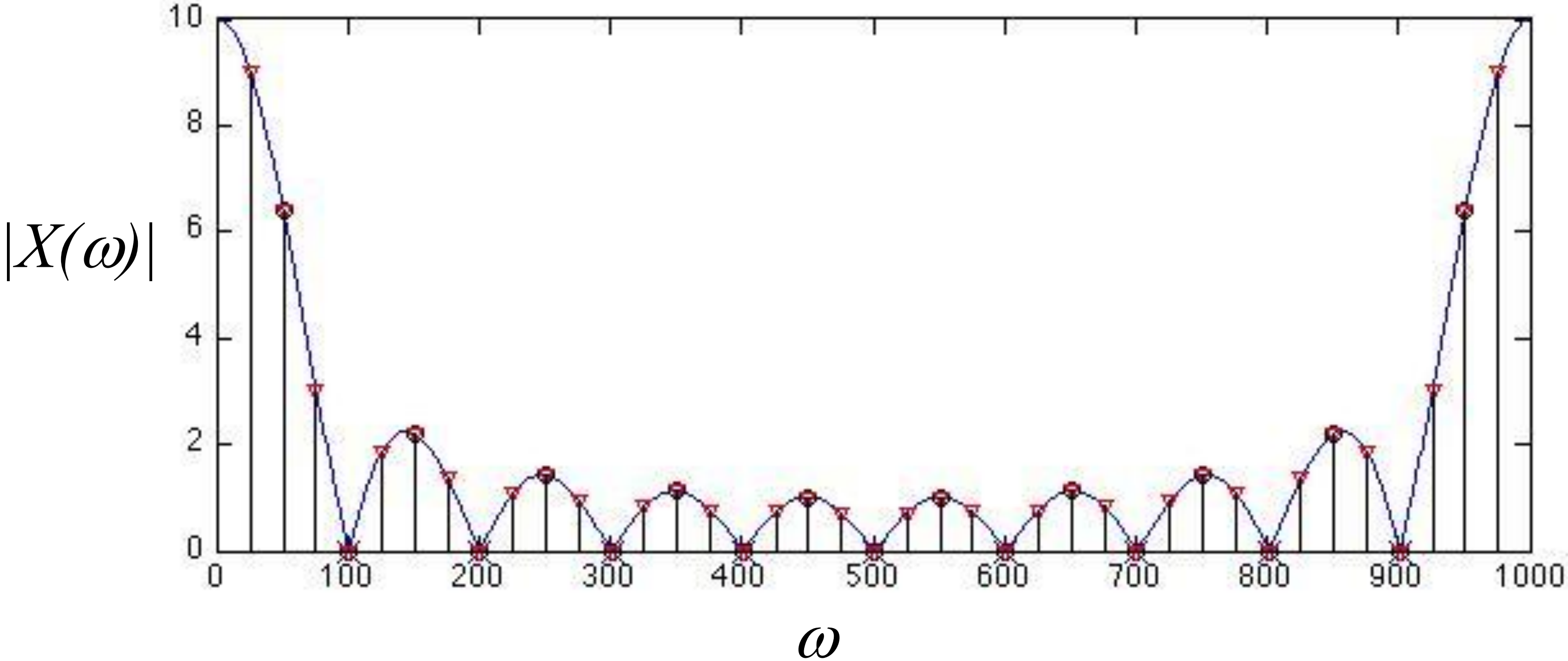
$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

$$X(2\pi k/N) = \frac{\sin(\pi k L/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N}$$

Example (cntd.)

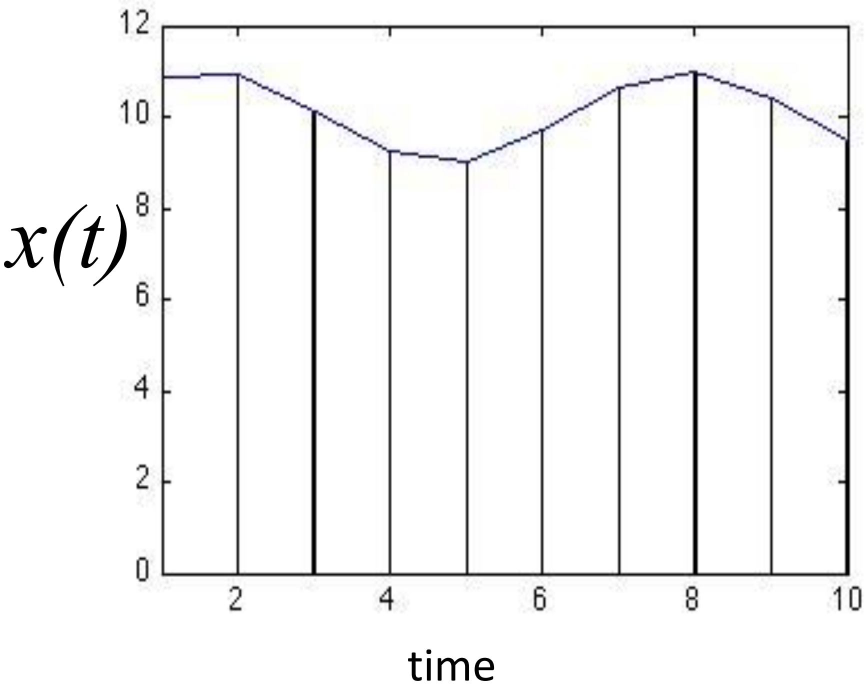


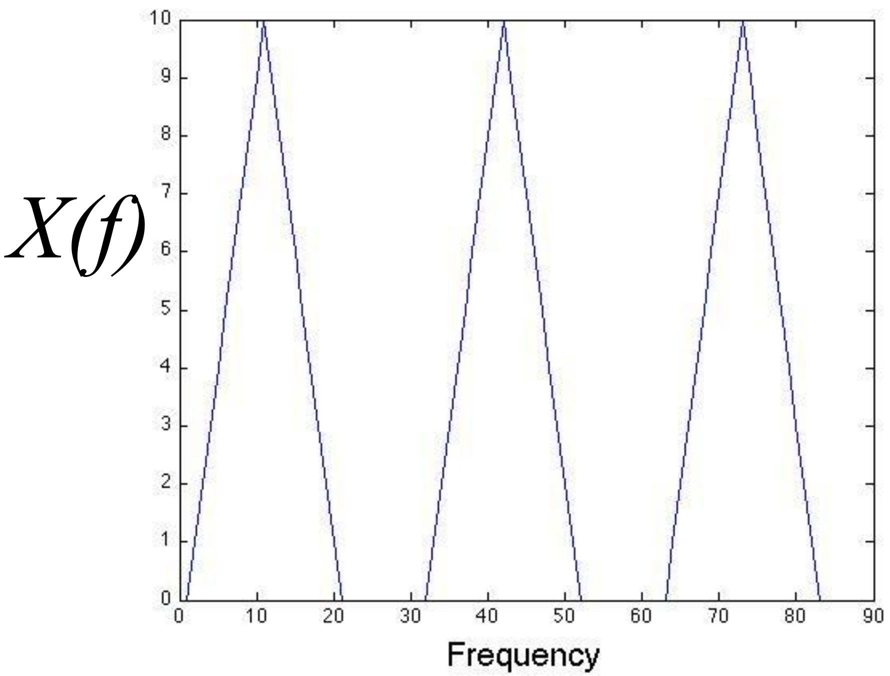
Example (cntd.)



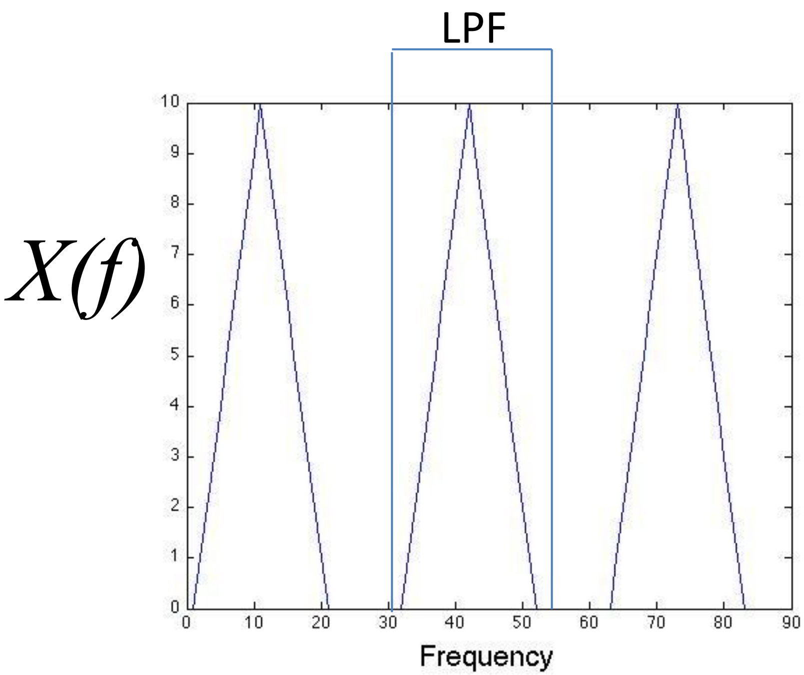
DTFT, N=L, N=2L, 4L

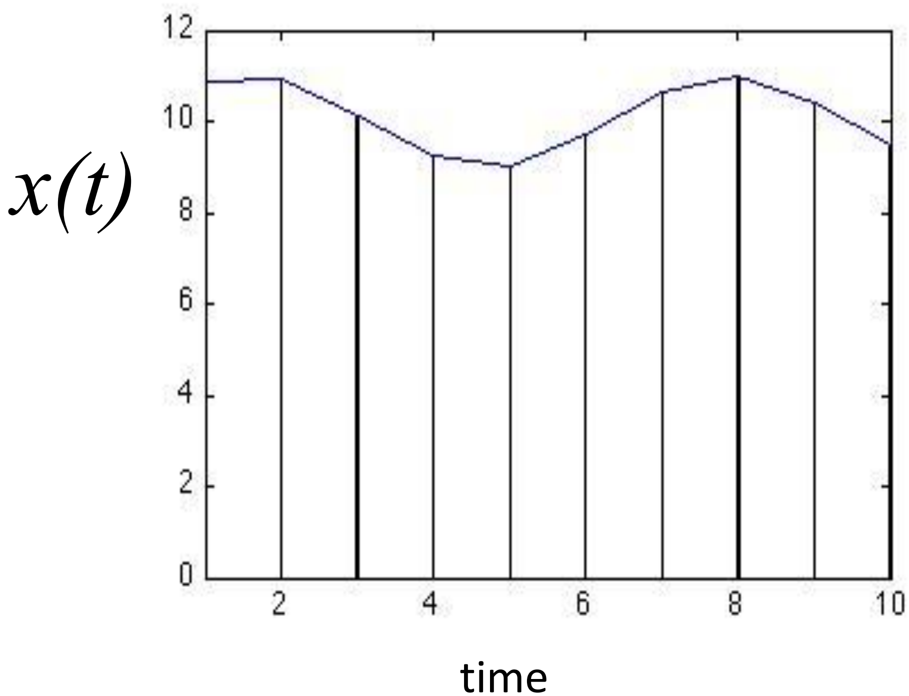
Time Domain Sampling



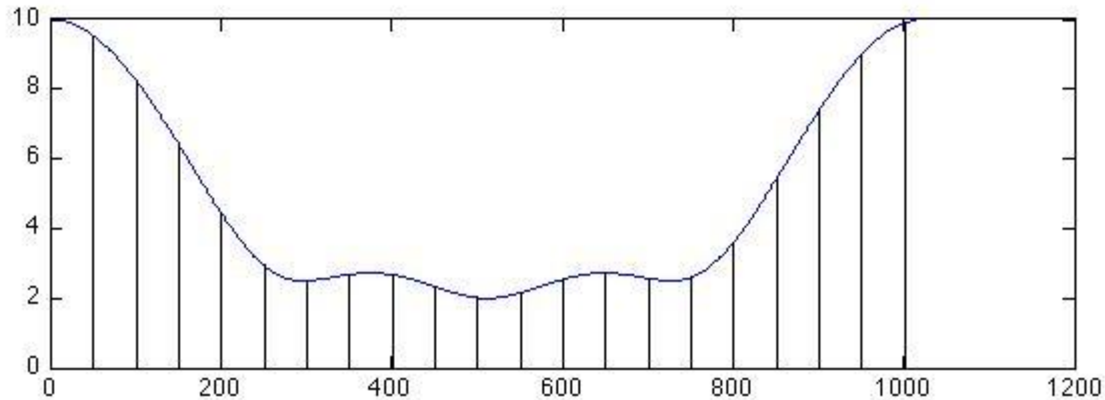


Time Domain Reconstruction



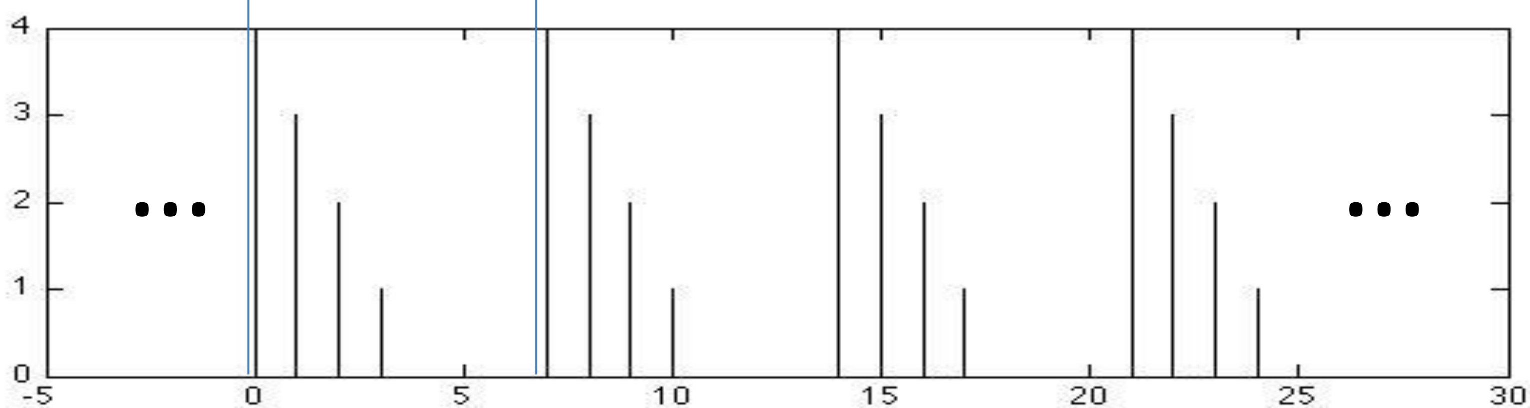


Frequency domain sampling



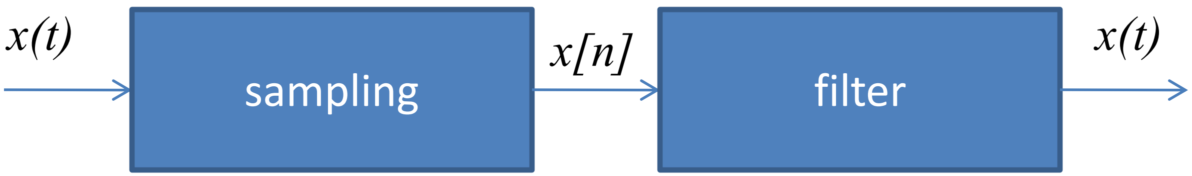
freq

Window 0…N-1

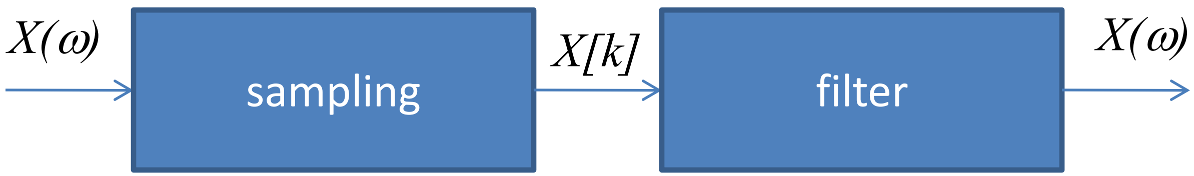


time

Analogy



$$H(f)=\begin{cases}1 & -f_{s/2} < f < f_{s/2} \\ 0 & \text{o.w.}\end{cases}$$



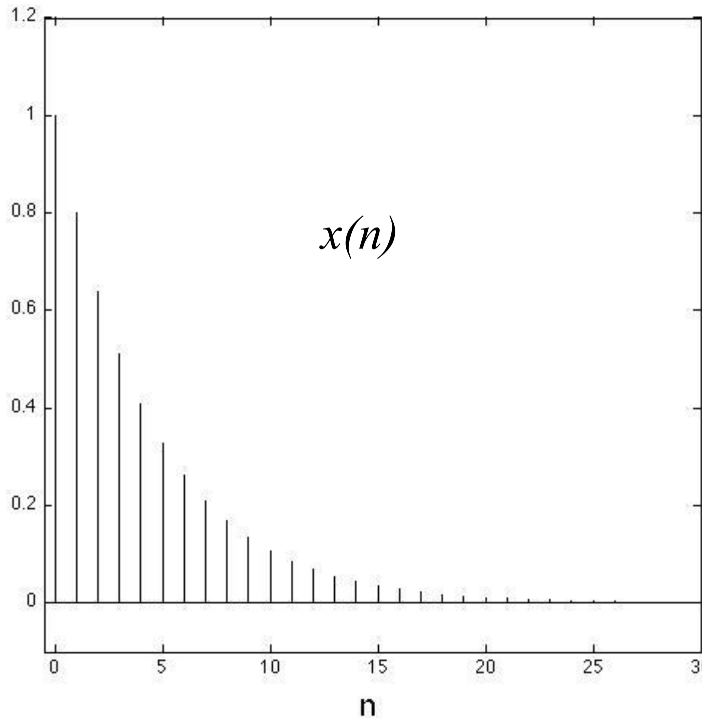
$$h(n)=\begin{cases}1 & 0 \le n \le N-1 \\ 0 & \text{o.w.}\end{cases}$$

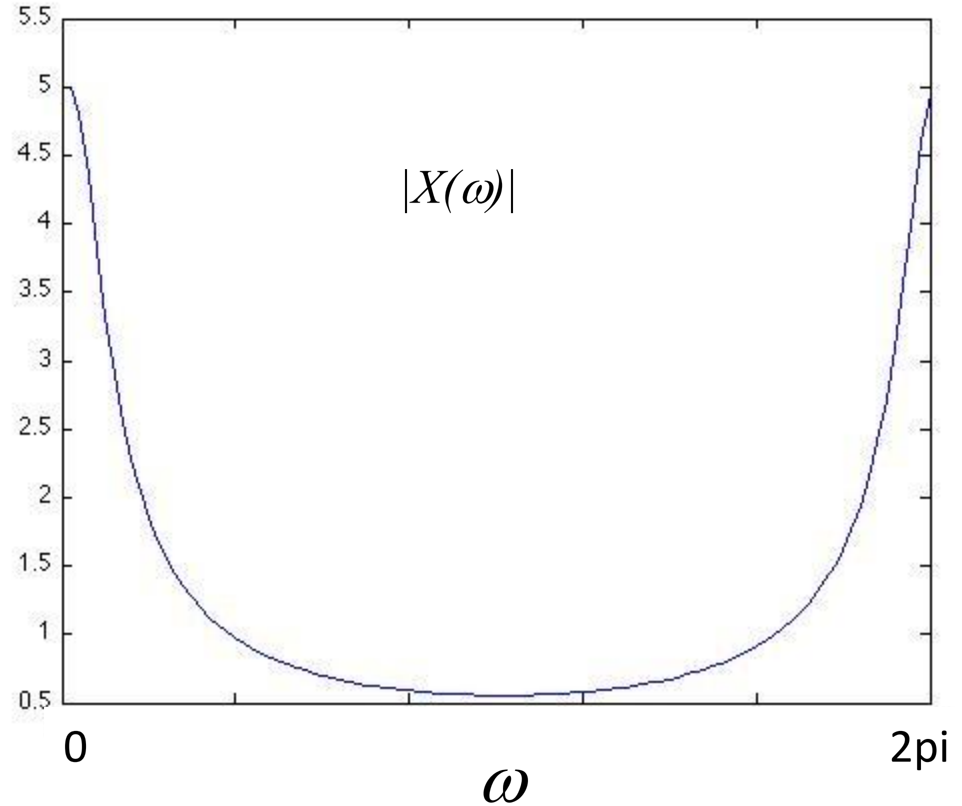
$$H(\omega)=\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Example

$$x(n) = a^n u(n) \qquad 0<a<1$$

Determine the reconstructed spectrum





$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1-ae^{-j\omega}}$$

Example (cntd.)

$$X(k) = \frac{1}{1 - ae^{-j2\pi k/N}}$$ Take IDFT to get $$x_p(n) = \frac{a^n}{1-a^N} \qquad 0 \le n \le N-1$$

If N is taken large enough, $x_p(n)$ will approach $x(n)$

To reconstruct the frequency domain, calculate

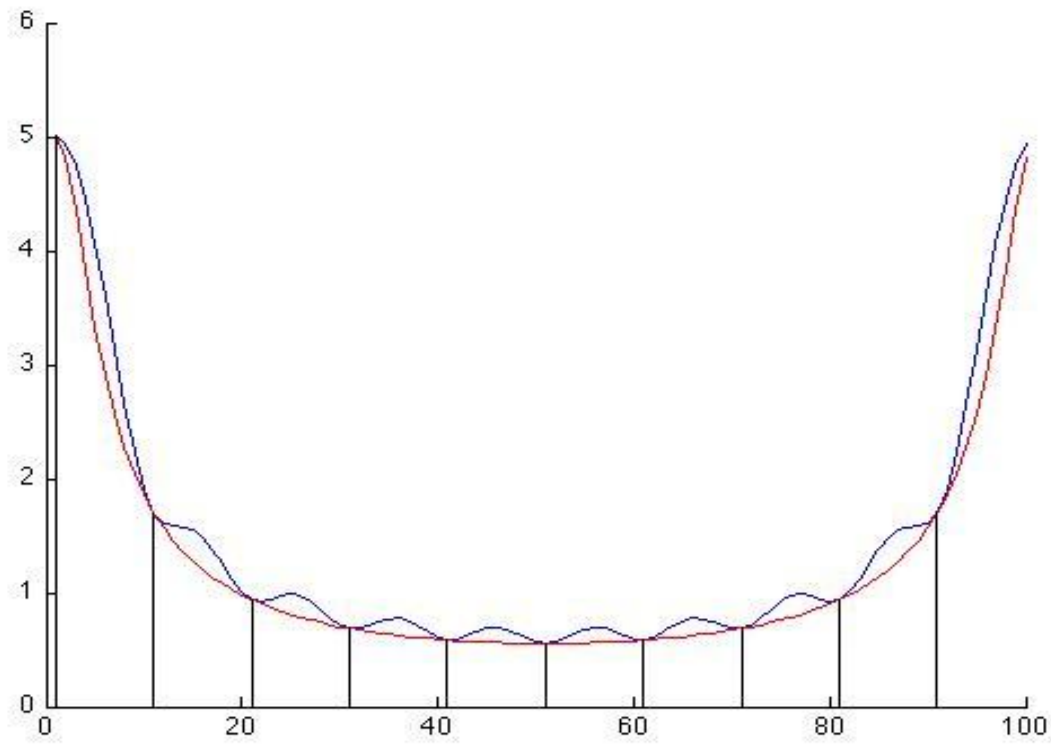
$$X(\omega) = \sum_{n=0}^{N-1} x_p(n) e^{-j\omega n} = \frac{1}{1-a^N} \frac{1 - a^N e^{-j\omega N}}{1 - ae^{-j\omega}}$$

If N is large, a^N will decrease and the error will decrease

Example (cntd.)

$|X(\omega)|$

Reconstructed spectrum



ω

DFT Properties

$$x(n) \leftrightarrow X(k)$$

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

$$W_N = e^{-j2\pi/N}$$

$$x(n+N) = x(n)$$

$$X(k+N) = X(k)$$

Symmetry Properties of DFT

- If $x(n)$ is real, then $X(k) = X^*(N-k)$
- More properties can be obtained for even, odd, imaginary,… signals

Parseval's Theorem

$$\sum_{n=0}^{N-1} x(n)y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)Y^*(k)$$

$$\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X(k)|^2$$

Linearity and Shift

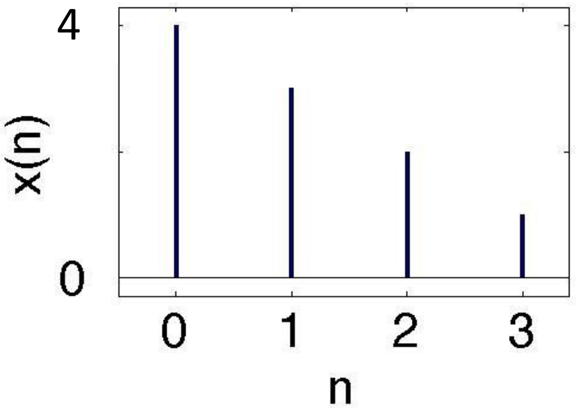
$$ax(n)+by(n)\overset{DFT}{\underset{N}{\leftrightarrow}} aX(k)+bY(k)$$

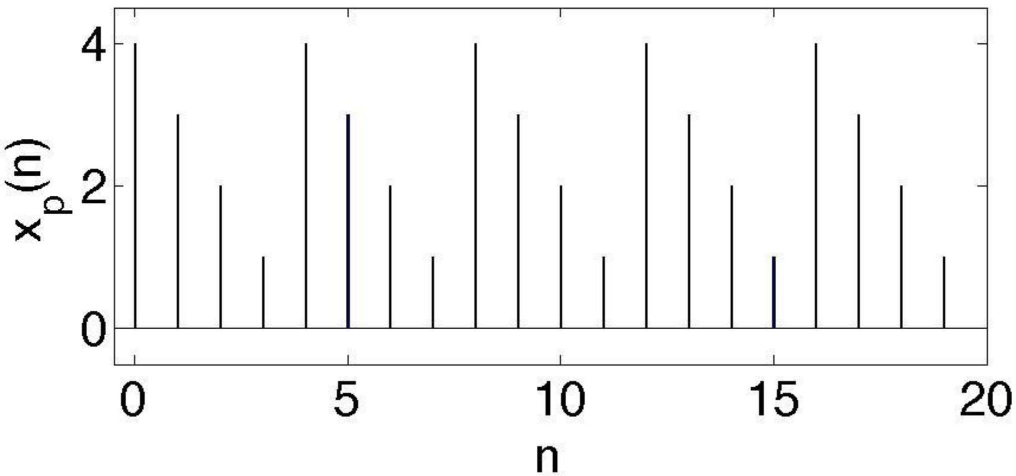
The N-point DFT of a finite duration sequence, x(n) of length L<N is equivalent to the N-point DFT of a periodic sequence $x_p(n)$ of period N

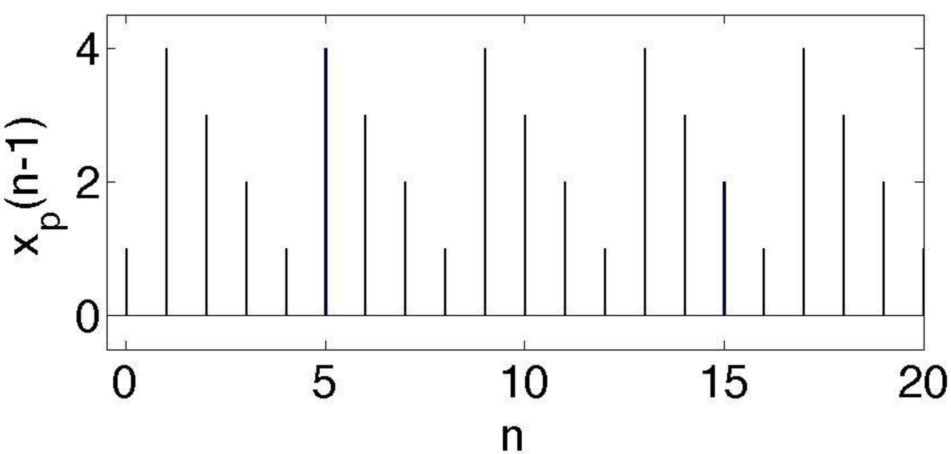
Any shift by m units, will be applied to $x_p(n)$ such that

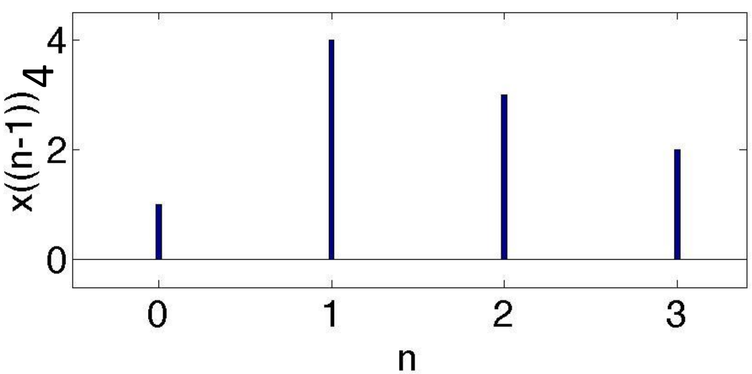
$$x_p(n-m)=\sum_{l=-\infty}^{\infty} x(n-m-lN)$$

Shift





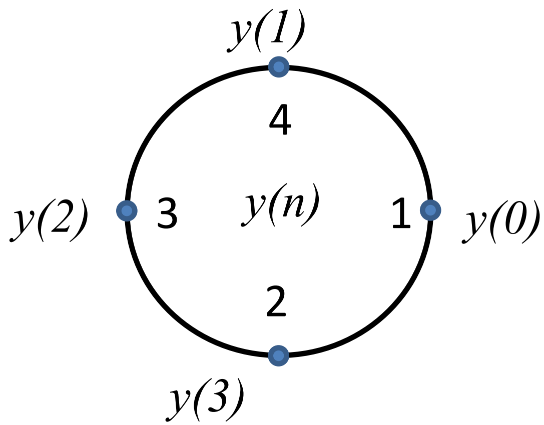


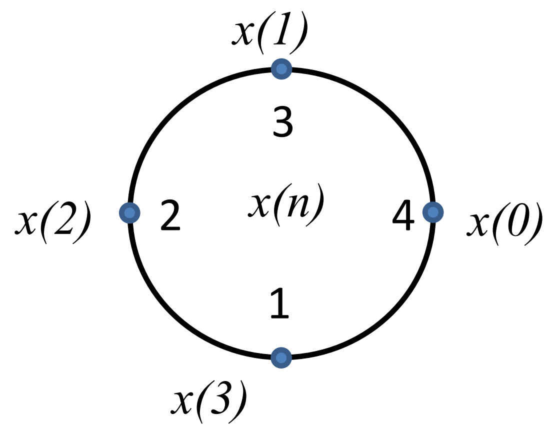


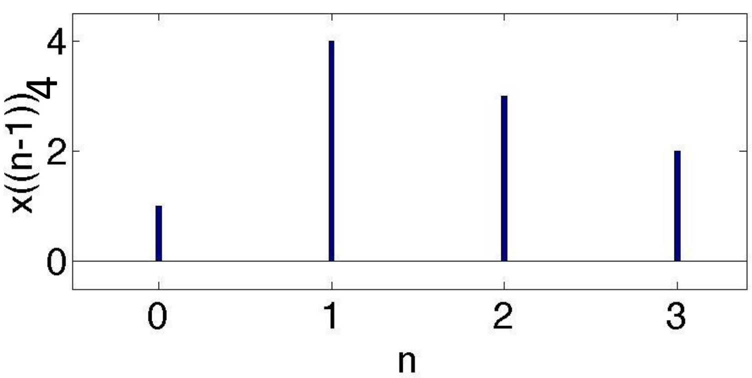
$$x((n-m))_N = x(n-m \bmod N)$$

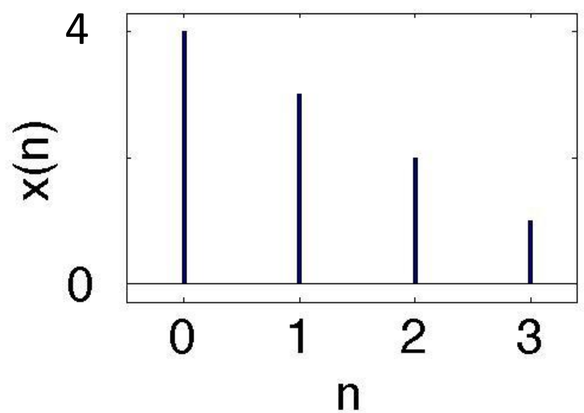
Circular Shift

$$y(n) = x((n-1))_4$$









Multiplication of 2 DFTs

In analog systems, using Fourier transform

If $$Y(f) = H(f)x(f)$$

Then $$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

Using the DFT

If $$Y(k) = X(k)H(k)$$

$$y(m) = \sum_{n=0}^{N-1} x(n)h((m-n))_N$$

Circular convolution

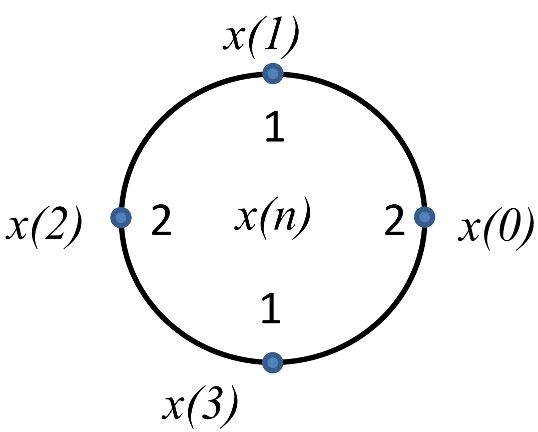
Note that both signals must have the same size N

Example

Perform circular convolution between the following 2 sequences:

$$x=[2\ 1\ 2\ 1\ ]\ h=[1\ 2\ 3\ 4]$$

$$y(m)=\sum_{n=0}^{4-1}x(n)h((m-n))_4$$



Example

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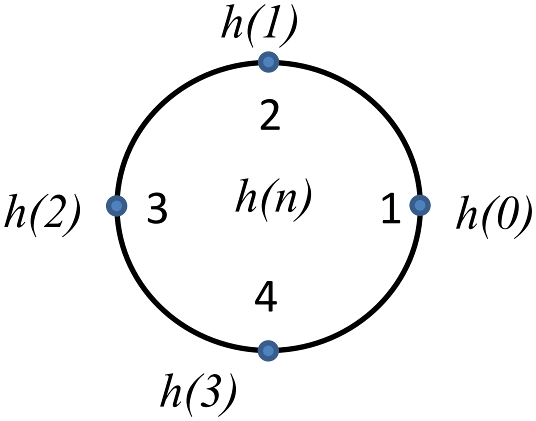
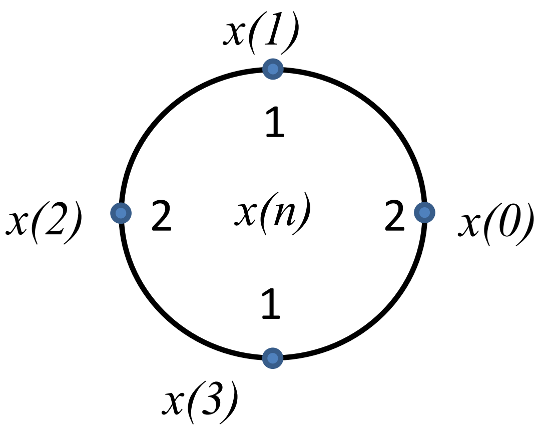
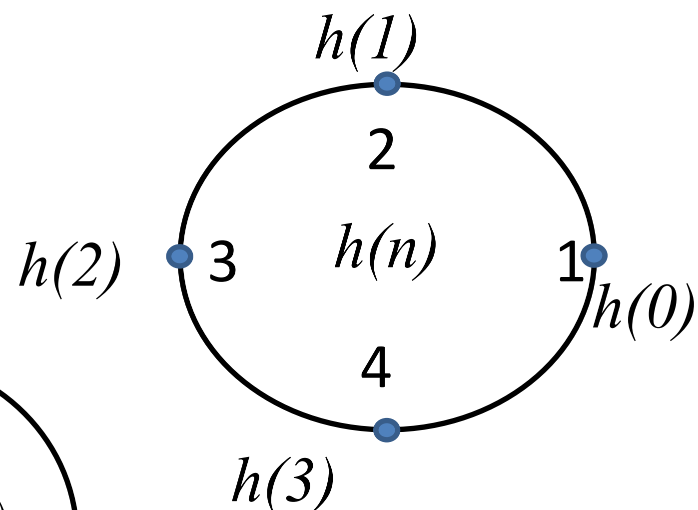


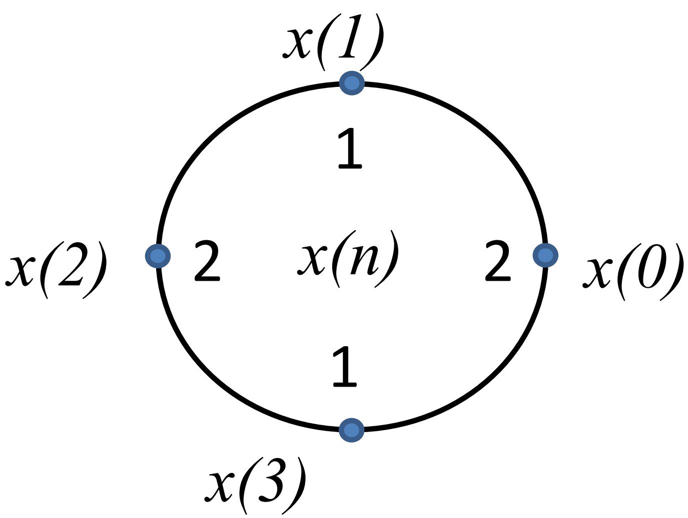


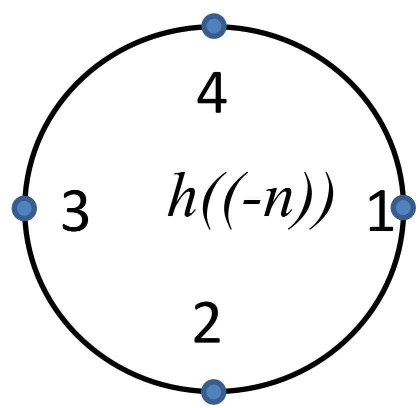
Example

Perform circular convolution between the following 2 sequences: x=[2 1 2 1] h=[1 2 3 4]

$$y(m) = \sum_{n=0}^{4-1} x(n)h((m-n))_4$$

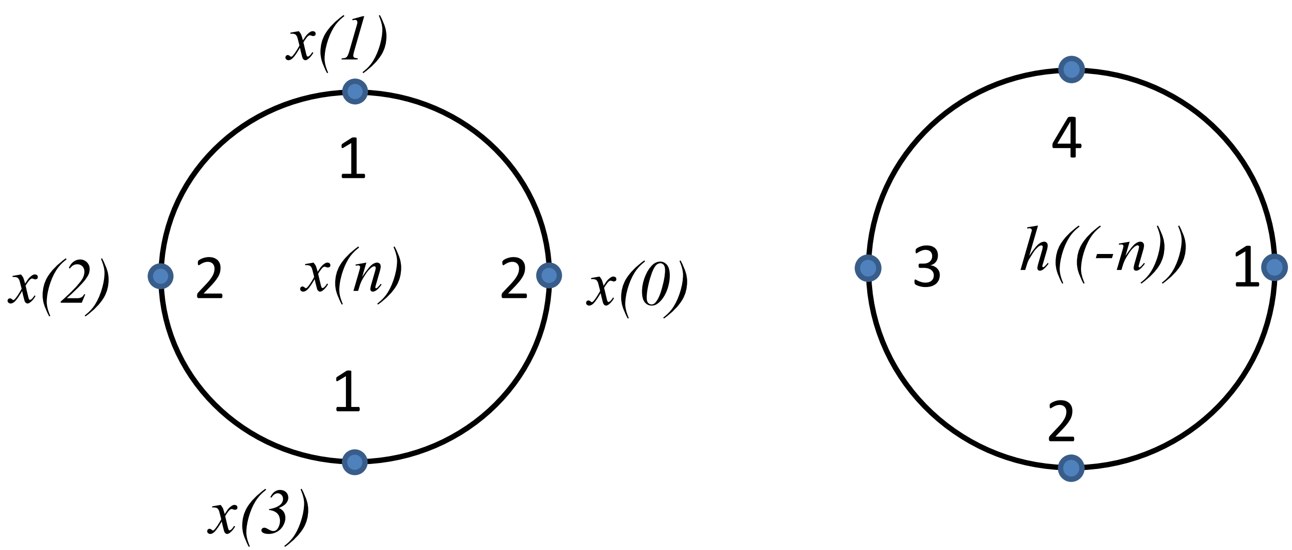






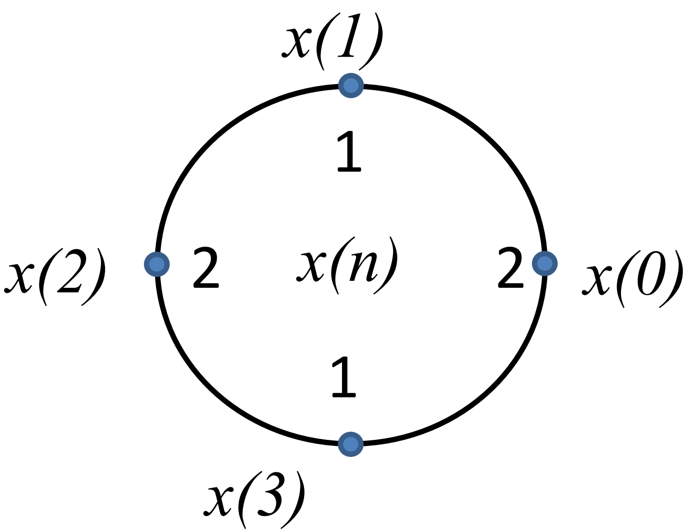
Example (cntd.)

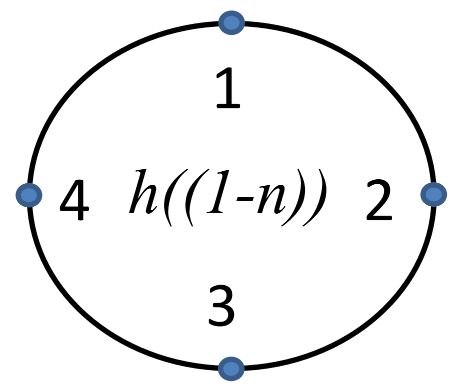
$$y(0) = \sum_{n=0}^{4-1} x(n)h((-n))_4 = 14$$



Example (cntd.)

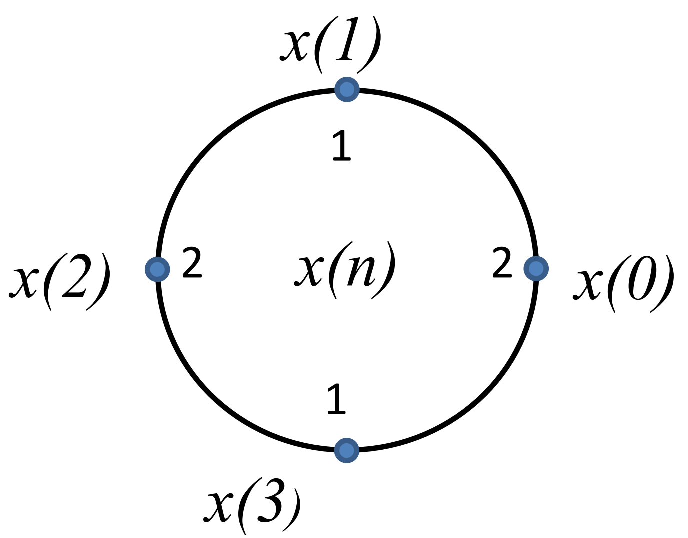
$$y(1) = \sum_{n=0}^{4-1} x(n)h((1-n))_4 = 16$$

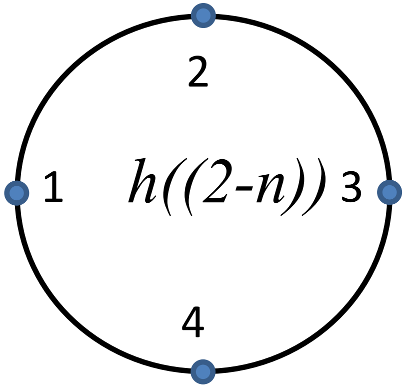




Example (cntd.)

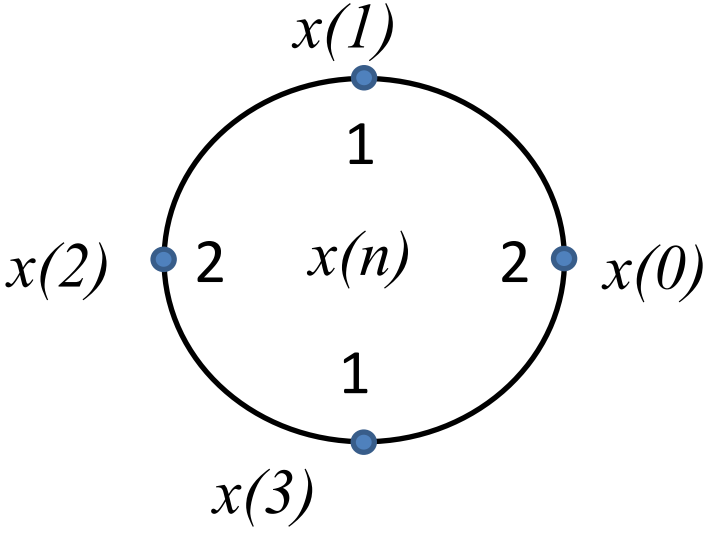
$$y(2)=\sum_{n=0}^{4-1}x(n)h((2-n))_4=14$$

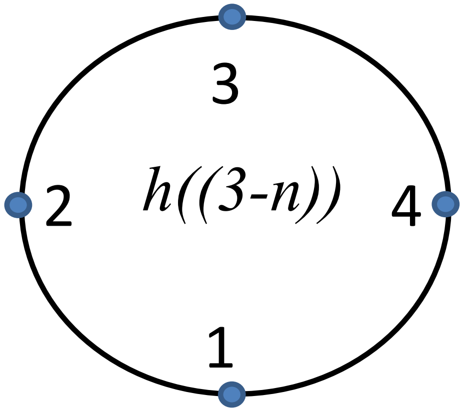




Example (cntd.)

$$y(3) = \sum_{n=0}^{4-1} x(n)h((1-n))_4 = 16$$





$y(n)$=[14 16 14 16]

Linear Filtering Based on DFT

$$Y(k) = X(k)H(k)$$

$$y(m) = \sum_{n=0}^{N-1} x(n)h((m-n))_N$$

- To perform circular convolution, x and h must be of the same size, N, and y is also of size N.
- y is NOT equal to the linear convolution of x and h
- How can we FORCE the circular convolution to calculate the linear convolution?

Linear Filtering Based on DFT

- Assume x is of length L and h is of length M
- The linear convolution output will be of length N=L+M-1
- In the case of linear convolution,

$$Y(\omega) = X(\omega)H(\omega)$$

$$y(m) = \sum_{n=0}^{L-1} x(n)h(m-n)$$

Linear Filtering Based on DFT

- The sequence $y(n)$ can be uniquely represented in the frequency domain by N=M+L-1 samples of its spectrum $Y(\omega)$

$$Y(k) = Y(\omega)\big|_{\omega=2\pi k/N} \qquad k = 0,1,\ldots,N-1$$

$$= X(\omega)H(\omega)\big|_{\omega=2\pi k/N} \qquad k = 0,1,\ldots,N-1$$

$$= X(k)H(k)\big|_{\omega=2\pi k/N} \qquad k = 0,1,\ldots,N-1$$

- Where $X(k)$ and $H(k)$ are the N-point DFTs of $x(n)$ and $h(n)$

Linear Filtering Based on DFT

- We PAD $x(n)$ and $h(n)$ with zeros to increase their duration to N
- Padding does not change the spectra $X(\omega)$ and $H(\omega)$
- By increasing the lengths of $x[n]$ and $h[n]$ to N and then circularly convolving the resulting sequences, we obtain the same result we would obtain with linear convolution

Example

- Perform Linear convolution using Circular convolution for $x(n)$=[1 2 2 1], $h(n)$=[1 2 3]
- L=3, M=4, use N=4+3-1=6

Do circular convolution between

$x(n)$=[1 2 2 1 0 0], $h(n)$=[1 2 3 0 0 0]

You can also get the DFT(x) and DFT(h) with N=6,

Calculate $Y(k)=X(k)H(K)$, N=6

Then $y(n)$=IDFT(Y)

Frequency Analysis Using DFT

- To compute the spectrum, the signal values for all time are required, which is non-practical.
- What are the implications of using a finite data record in frequency analysis using the DFT?
- Assume a band-limited, analog signal with BW=B, sampling rate $F_s=1/T>2B$, and L total samples.

Frequency Analysis Using DFT

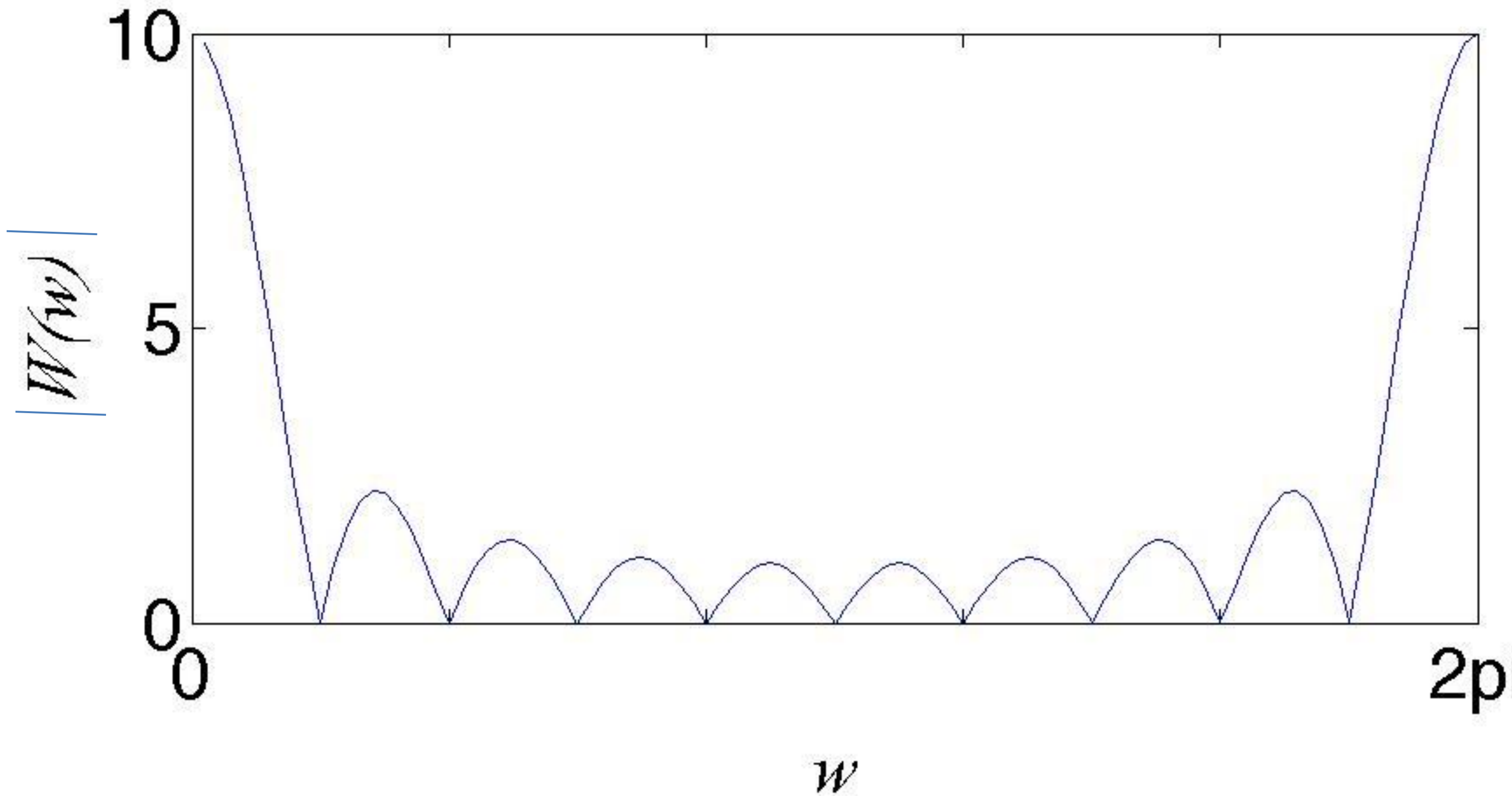
Assume $x(n)$ is the signal to be analyzed, limiting the number of samples to L is equivalent to multiplying $x(n)$ by a rectangular window $w(n)$ of length L

$$\tilde{x}(n) = x(n)w(n)$$

$$w(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{o.w.} \end{cases}$$

$$W(\omega) = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

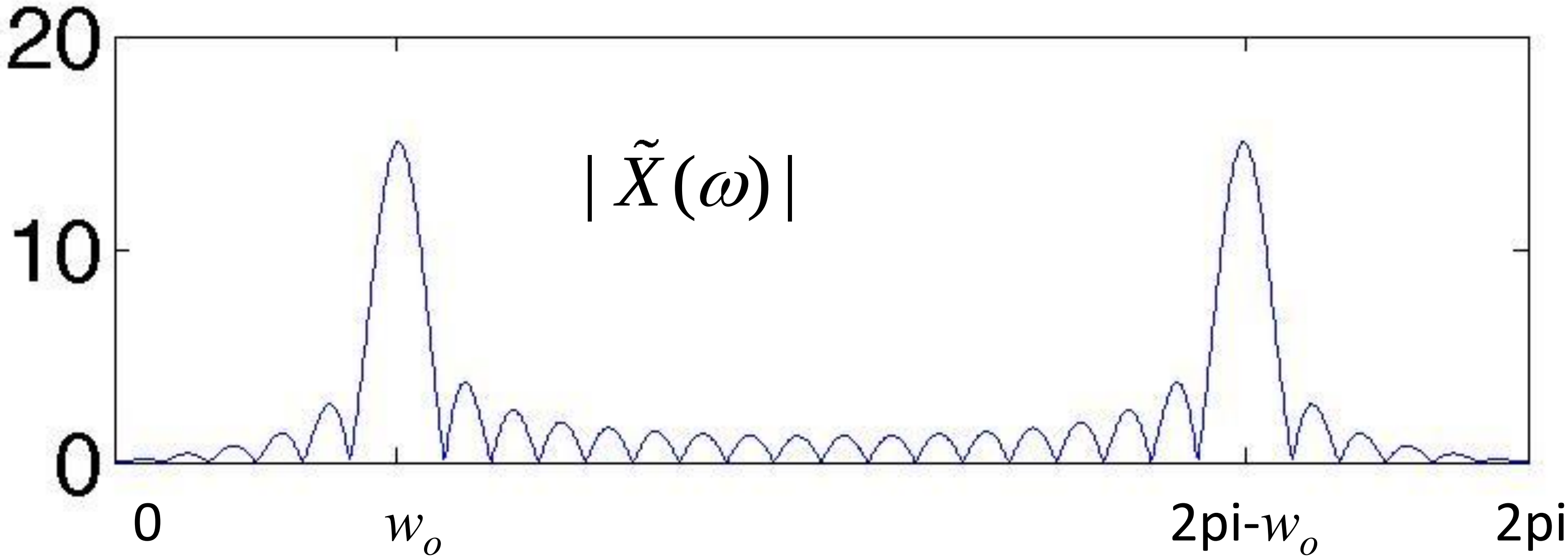
Frequency Analysis Using DFT



Frequency Analysis Using DFT

- Assume that the input signal, $x(n) = \cos(w_o n)$

$$\tilde{X}(\omega) = 1/2\left[W(\omega - \omega_o) + W(\omega + \omega_o)\right]$$



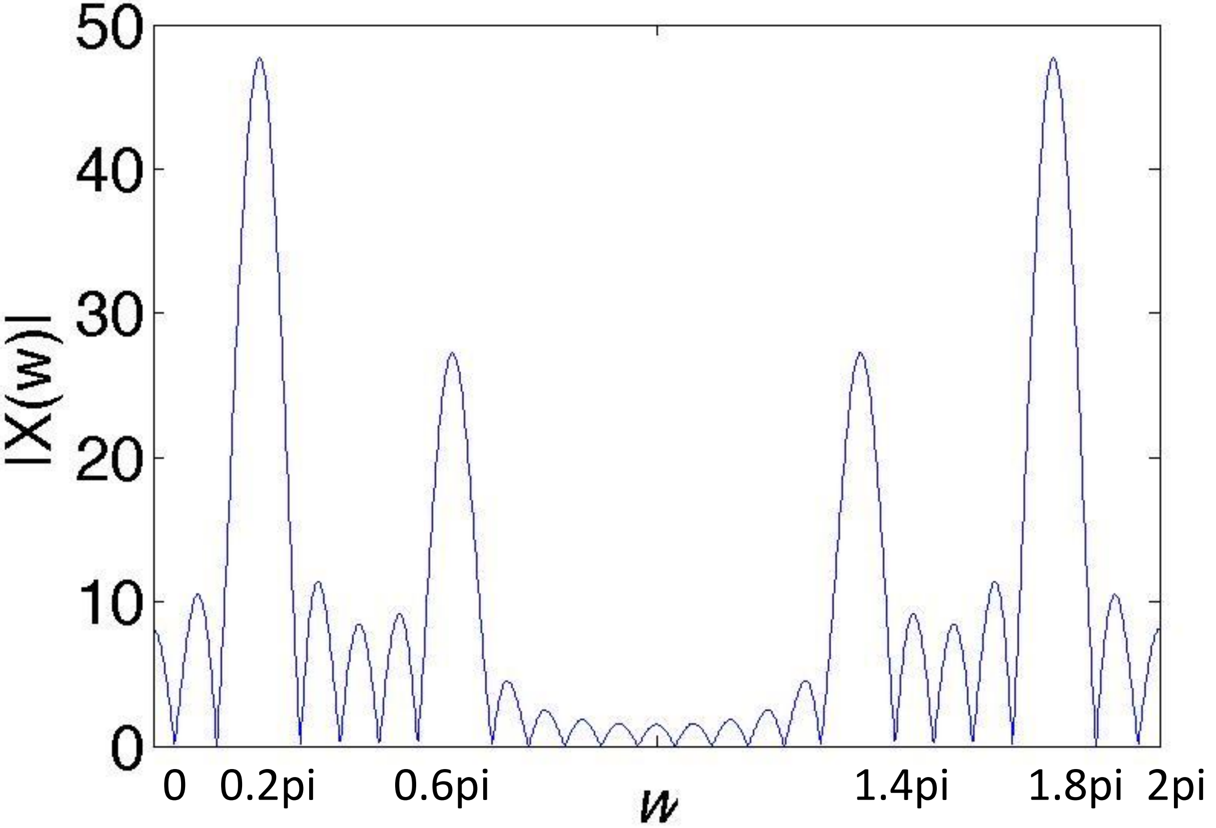
Frequency Analysis Using DFT

- To compute $\tilde{X}(\omega)$ we use the DFT, by padding $\tilde{x}(n)$ with N-L zeros to compute the N-point DFT of the L point sequence
- Notice that the windowed spectrum is not localized to a single frequency but the power "leaked out" into the entire frequency range
- Leakage not only distort the spectrum but decreases resolution as well

Example

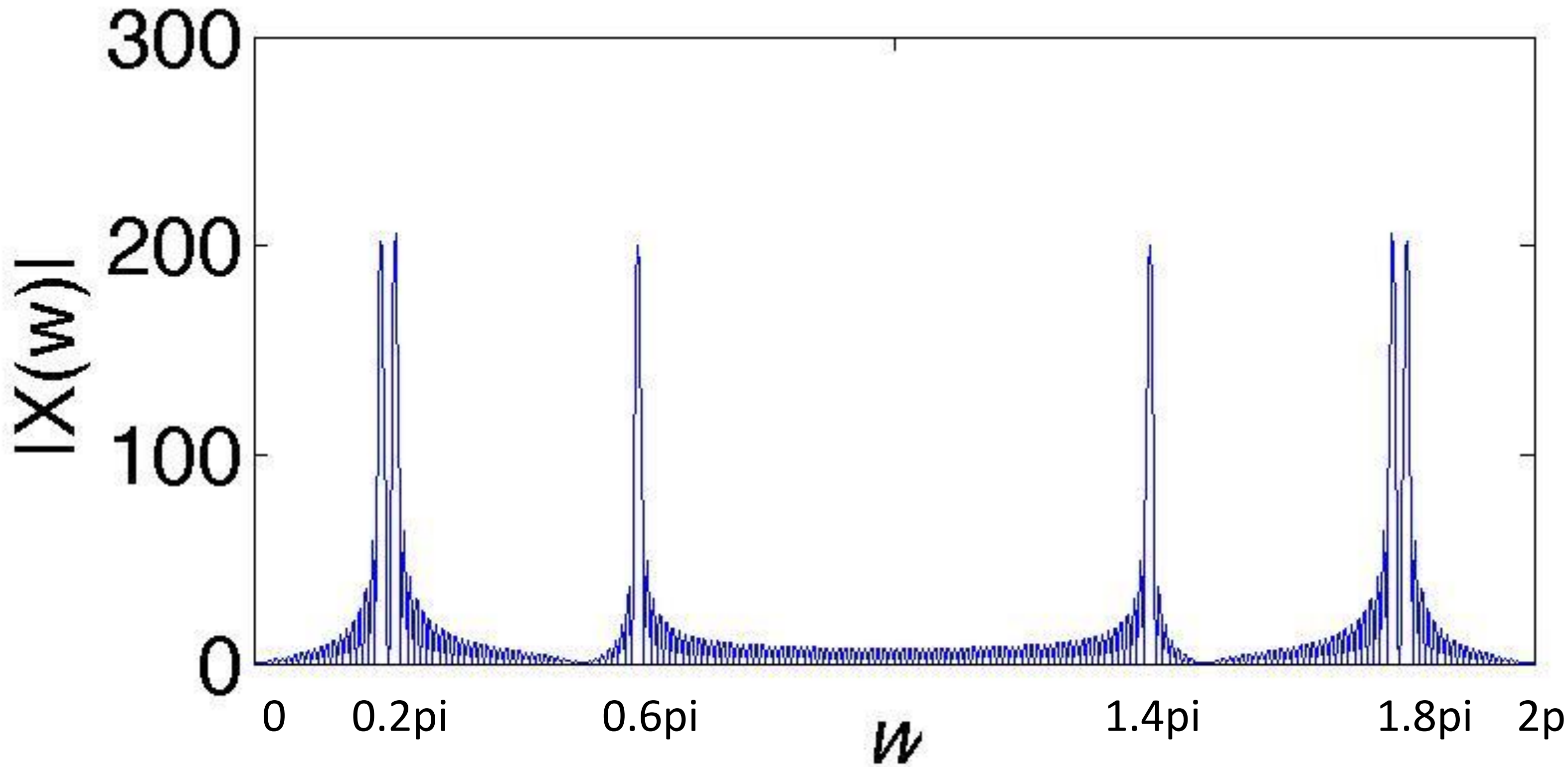
$$x(n) = \cos(\omega_0 n) + \cos(\omega_1 n) + \cos(\omega_2 n)$$

$$\omega_0 = 0.2\pi, \omega_1 = 0.22\pi, \omega_2 = 0.6\pi$$



L=25, N=1000

Example (cntd.)



L=200,N=1000

Frequency Resolution of DFT

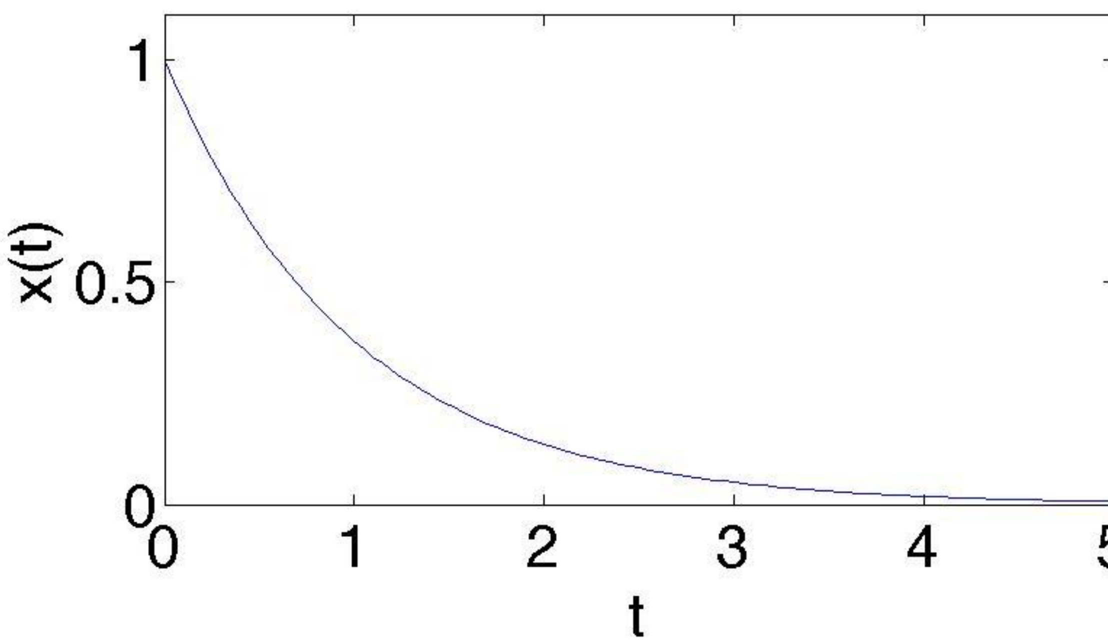
- The spectrum of the rectangular window sequence has its first zero at $\omega = 2\pi / L$
- If $|\omega_1 - \omega_2| < 2\pi / L$ the 2 window functions $W(\omega - \omega_1)$ and $W(\omega - \omega_2)$ overlap and thus the 2 spectral lines are not distinguishable
- $2\pi/L$ defines the frequency resolution of the DFT
- Increasing N increase the visibility of the spectrum already defined by L

Example

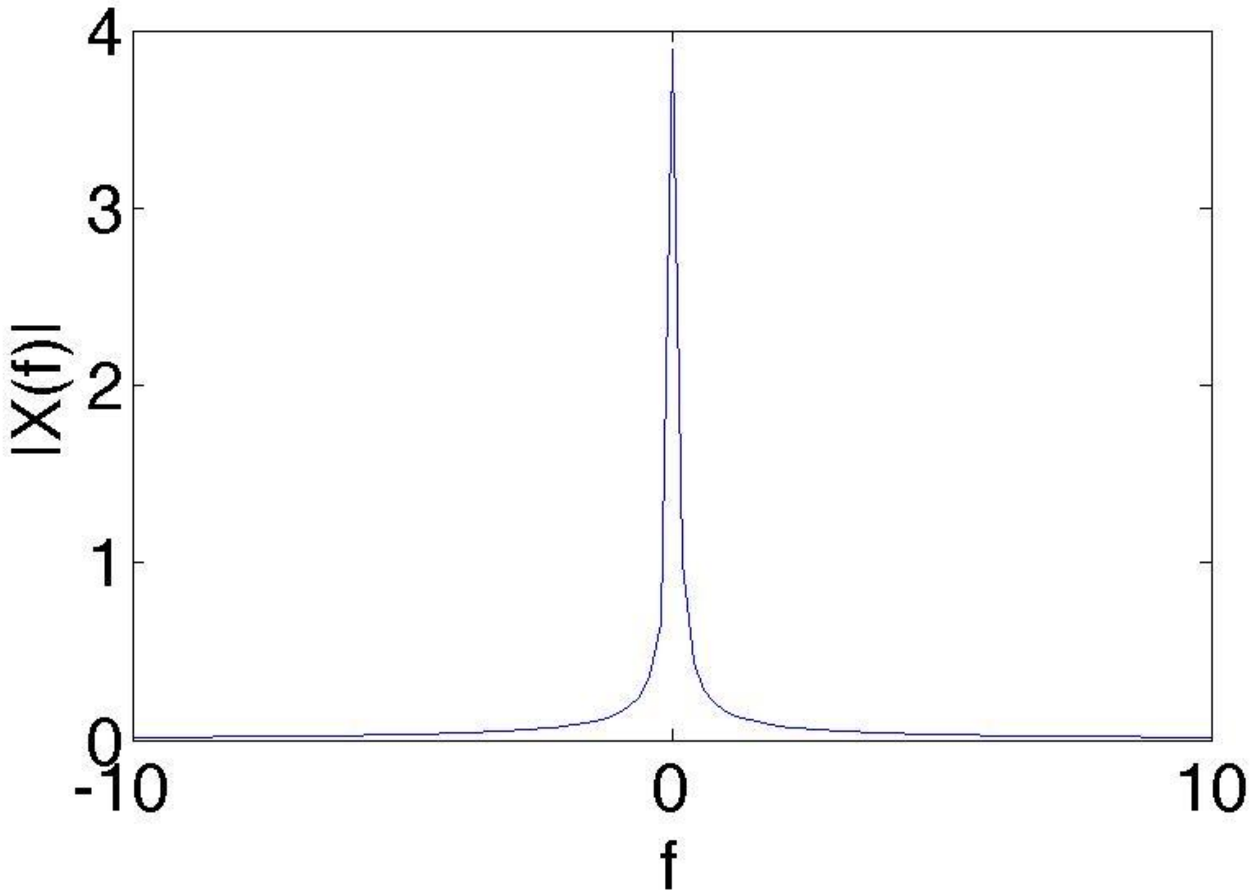
$$x_a(t) = \begin{cases} e^{-t}, & t \geq 0 \\ 0 & t<0 \end{cases}$$

Is sampled at the rate Fs=20 samples per sec, and a block of length 100 samples is used to estimate the spectrum. Compare the spectrum of the truncated discrete signal to the spectrum of the analog signal.

$$X_a(F) = \frac{1}{1+j2\pi F}$$

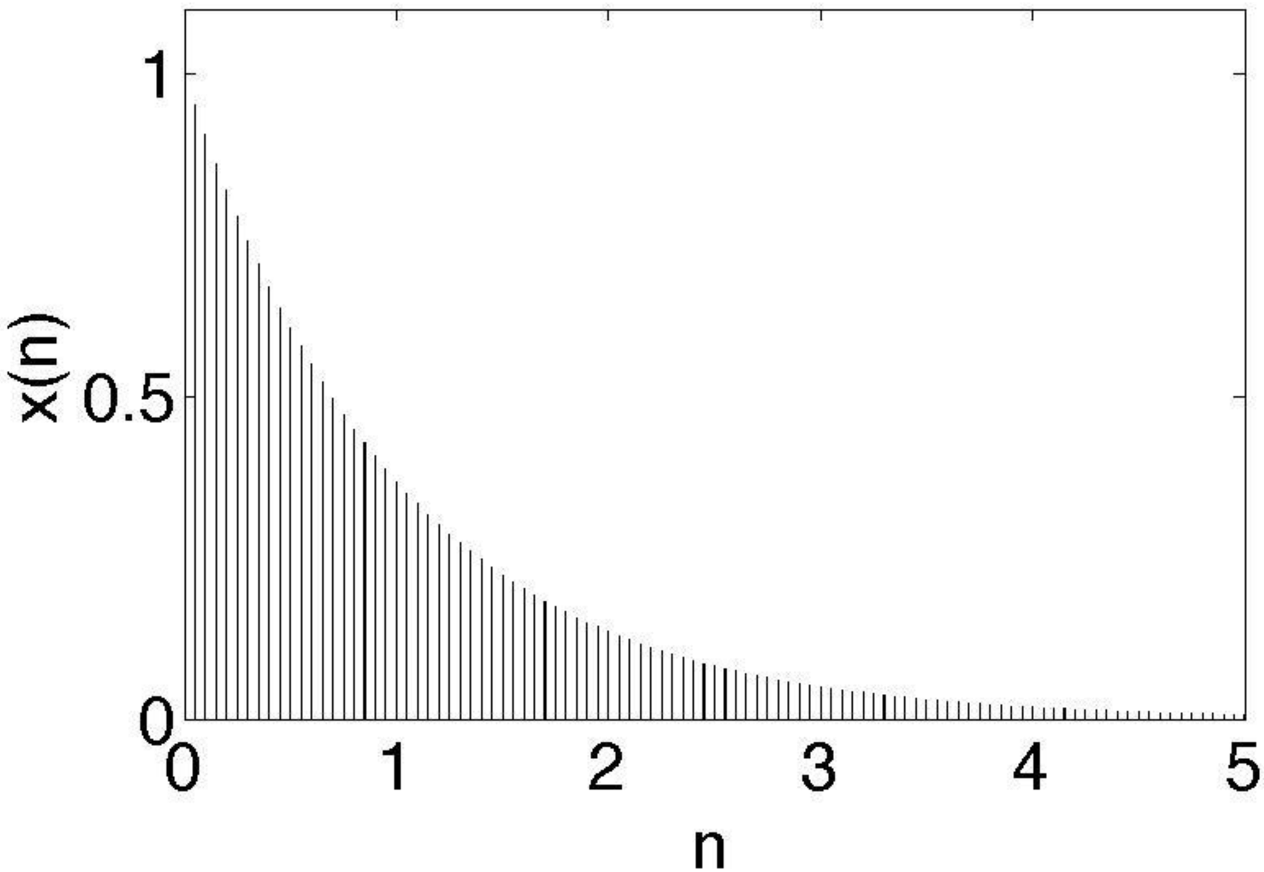


Example (cntd.)



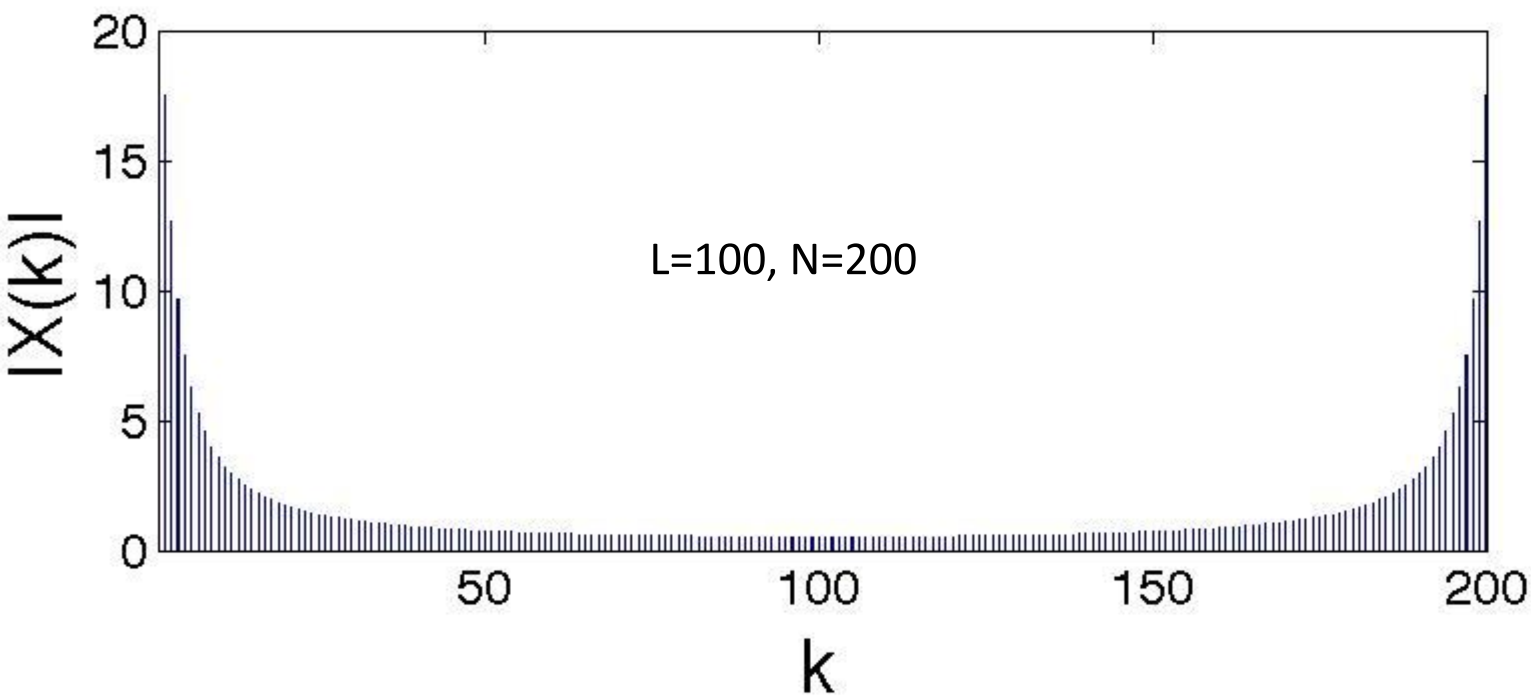
Example (cntd.)

$$x(n) = e^{-nT_s} = e^{-n/20} = (0.95)^n, \quad \mathrm{n} \geq 0$$



Example (cntd.)

- We would like to use L samples only of the discrete signal to compute the spectrum of the signal



Example (cntd.)