Reflection and Transmission

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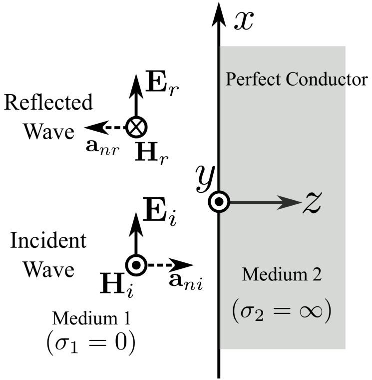
Outline

Reference

D. K. Cheng, Field and Wave Electromagnetics, 2nd edition. Reading, Mass: Addison-Wesley, 1989.

Outline

Normal Incidence at a Plane Conducting Boundary



$$\mathbf{E}_i(z) = \mathbf{a}_x E_{0i} e^{-j\beta_1 z},$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{0i}}{\eta_1} e^{-j\beta_1 z}$$

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{0r} e^{j\beta_1 z},$$

$$\mathbf{H}_r(z) = -\mathbf{a}_y \frac{E_{0r}}{\eta_1} e^{j\beta_1 z}$$

where $\eta_1 = \sqrt{\frac{\mu}{\epsilon}}$, $\beta_1 = \omega\sqrt{\mu\epsilon}$.

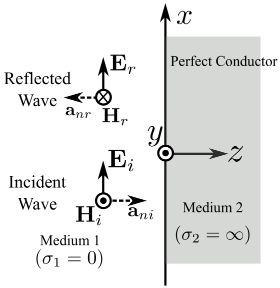
- In medium 1 ($z < 0$),

$$\mathbf{E}_1(z) = \hat{\mathbf{a}}_x \left(E_{0i} e^{-j\beta_1 z} + E_{0r} e^{+j\beta_1 z} \right),$$

$$\mathbf{H}_1(z) = \frac{\hat{\mathbf{a}}_y}{\eta_1} \left(E_{0i} e^{-j\beta_1 z} - E_{0r} e^{+j\beta_1 z} \right)$$

- In medium 2 ($z > 0$), $\mathbf{E}_2 = \mathbf{0}$, $\mathbf{H}_2 = \mathbf{0}$

Normal Incidence at a Plane Conducting Boundary



$$\mathbf{E}_1(z) = \hat{\mathbf{a}}_x \left(E_{0i} e^{-j\beta_1 z} + E_{0r} e^{+j\beta_1 z} \right),$$

$$\mathbf{H}_1(z) = \frac{\hat{\mathbf{a}}_y}{\eta_1} \left(E_{0i} e^{-j\beta_1 z} - E_{0r} e^{+j\beta_1 z} \right)$$

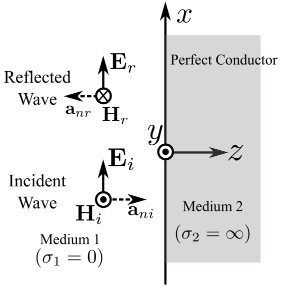
- Boundary Conditions: Continuity of the tangential electric field at $z = 0$,

$$\mathbf{E}_1(0) = \hat{\mathbf{a}}_x (E_{0i} + E_{0r}) = \mathbf{0}, \quad \Rightarrow \quad E_{0r} = -E_{0i}$$

- The fields in medium 1 $(z < 0)$,

$$\mathbf{E}_1(z) = -2j\hat{\mathbf{a}}_x E_{0i} \sin(\beta_1 z),$$

$$\mathbf{H}_1(z) = 2\frac{\hat{\mathbf{a}}_y}{\eta_1} E_{0i} \cos(\beta_1 z)$$



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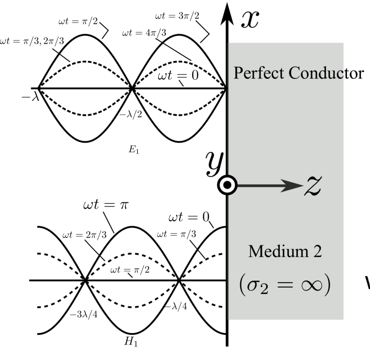
$$\mathbf{H}_1(z) = 2\frac{\hat{\mathbf{a}}_y}{\eta_1} E_{0i} \cos(\beta_1 z)$$

- Instantaneous fields in medium 1 ($z < 0$),

$$\mathbf{E}_1(z,t) = \Re\left\{\mathbf{E}_1(z)\, e^{j\omega t}\right\} = 2\hat{\mathbf{a}}_x E_{0i} \sin(\beta_1 z) \sin(\omega t),$$

$$\mathbf{H}_1(z,t) = \Re\left\{\mathbf{H}_1(z)\, e^{j\omega t}\right\} = 2\frac{\hat{\mathbf{a}}_y}{\eta_1} E_{0i} \cos(\beta_1 z) \cos(\omega t)$$

Normal Incidence at a Plane Conducting Boundary



Zeros of $\mathbf{E}_1(z,t)$ / Maxima of $\mathbf{H}_1(z,t)$ } at $\beta_1 z = -n\pi$ or $z = -n\lambda/2$

Maxima of $\mathbf{E}_1(z,t)$ / Zero of $\mathbf{H}_1(z,t)$ } at $\beta_1 z = -\dfrac{(2n+1)\pi}{2}$ or $z = -(2n+1)\lambda/4$

where $n = 0,\ 1,\ 2, \cdots$

- Instantaneous fields in medium 1 ($z < 0$),

$$\mathbf{E}_1(z,t) = \Re\left\{\mathbf{E}_1(z)\,e^{j\omega t}\right\} = 2\hat{\mathbf{a}}_x E_{0i}\sin(\beta_1 z)\sin(\omega t),$$

$$\mathbf{H}_1(z,t) = \Re\left\{\mathbf{H}_1(z)\,e^{j\omega t}\right\} = 2\frac{\hat{\mathbf{a}}_y}{\eta_1}E_{0i}\cos(\beta_1 z)\cos(\omega t)$$

Example

Example

A y-polarized uniform plane wave ($\mathbf{E}_i$, $\mathbf{H}_i$) with a frequency 100 (MHz) propagates in air in the $+x$ direction and impinges normally on a perfectly conducting plane at $x = 0$. Assuming the amplitude of $\mathbf{E}_i$ to be 6 mV/m, write the phasor and instantaneous expressions for,

1. $\mathbf{E}_i$ and $\mathbf{H}_i$ of the incident wave,
2. $\mathbf{E}_r$ and $\mathbf{H}_r$ of the reflected wave, and
3. $\mathbf{E}_1$ and $\mathbf{H}_1$ of the total wave in air.
4. Determine the location nearest to the conducting plane where E_1 is zero.

Outline

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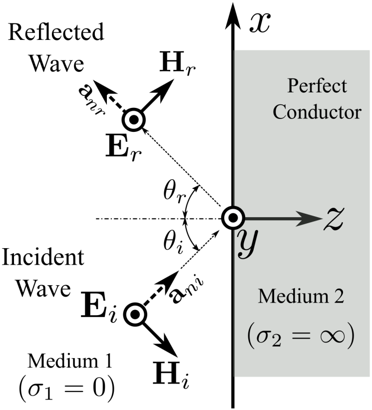
Perpendicular Polarization



- Incident wave $(z < 0)$

$$\mathbf{E}_i = \mathbf{a}_y E_{0i} e^{-j\beta_1 \mathbf{a}_{ni} \cdot \mathbf{R}}$$

$$\mathbf{a}_{ni} = \sin\theta_i \mathbf{a}_x + \cos\theta_i \mathbf{a}_z,$$

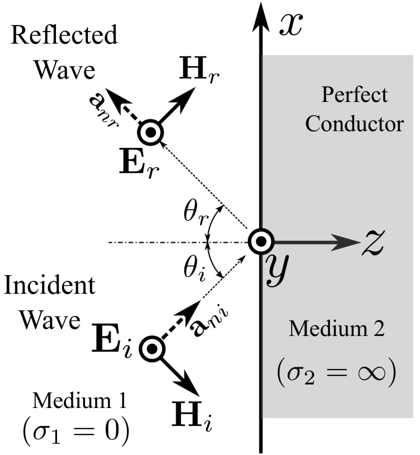
$$\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z,$$

$$\mathbf{E}_i(x, z) = \mathbf{a}_y E_{0i} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_i = \frac{1}{\eta_1} \mathbf{a}_{ni} \times \mathbf{E}_i$$

$$\mathbf{H}_i(x, z) = \frac{E_{0i}}{\eta_1} (\sin\theta_i \mathbf{a}_z - \cos\theta_i \mathbf{a}_x) e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

Perpendicular Polarization



- Reflected wave $(z < 0)$

$$\mathbf{E}_r = \mathbf{a}_y E_{0r} e^{-j\beta_1 \mathbf{a}_{nr} \cdot \mathbf{R}}$$

$$\mathbf{a}_{nr} = \sin\theta_r \mathbf{a}_x - \cos\theta_r \mathbf{a}_z,$$

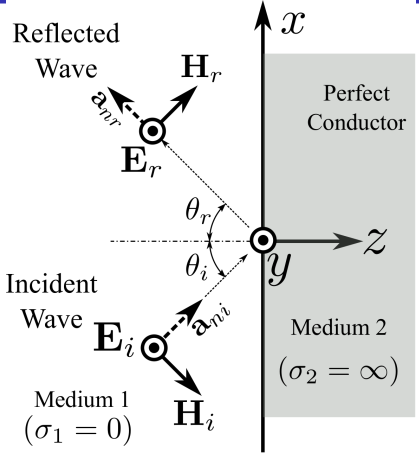
$$\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z,$$

$$\mathbf{E}_r(x,z) = \mathbf{a}_y E_{0r} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

$$\mathbf{H}_r = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r$$

$$\mathbf{H}_r(x,z) = \frac{E_{0r}}{\eta_1}(\sin\theta_r \mathbf{a}_z + \cos\theta_r \mathbf{a}_x)\, e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Perpendicular Polarization



- Total Electric Field $(z < 0)$

$$\mathbf{E}_1(x,z) = \mathbf{E}_i(x,z) + \mathbf{E}_r(x,z)$$

$$\mathbf{E}_i(x,z) = \mathbf{a}_y E_{0i} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{E}_r(x,z) = \mathbf{a}_y E_{0r} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

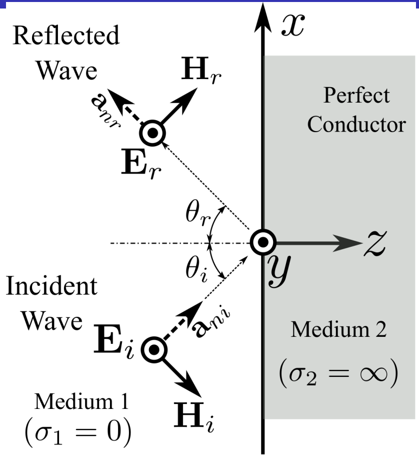
- Boundary Conditions at $z = 0$,

$$\mathbf{E}_1(x,0)\cdot\mathbf{a}_y = 0$$

$$E_{0i}e^{-j\beta_1 x\sin\theta_i} + E_{0r}e^{-j\beta_1 x\sin\theta_r} = 0$$

$$\boxed{\theta_r = \theta_i, \qquad E_{0r} = -E_{0i}}$$

Perpendicular Polarization



$$\mathbf{E}_i(x,z) = \mathbf{a}_y E_{0i} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{E}_r(x,z) = -\mathbf{a}_y E_{0i} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)},$$

$$\mathbf{H}_i(x,z) = \frac{E_{0i}}{\eta_1}(\sin\theta_i \mathbf{a}_z - \cos\theta_i \mathbf{a}_x) \times e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

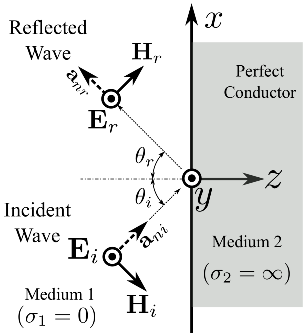
$$\mathbf{H}_r(x,z) = -\frac{E_{0i}}{\eta_1}(\sin\theta_i \mathbf{a}_z + \cos\theta_i \mathbf{a}_x) \times e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\mathbf{E}_1(x,z) = \mathbf{E}_i(x,z) + \mathbf{E}_r(x,z) = -2j\mathbf{a}_y E_{0i}\sin(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i}$$

$$\mathbf{H}_1(x,z) = \mathbf{H}_i(x,z) + \mathbf{H}_r(x,z)$$

$$= -\frac{2E_{0i}}{\eta_1}\left[j\sin\theta_i \sin(\beta_1 z\cos\theta_i)\,\mathbf{a}_z + \cos\theta_i\cos(\beta_1 z\cos\theta_i)\,\mathbf{a}_x\right] e^{-j\beta_1 x\sin\theta_i}$$

Perpendicular Polarization



$$\mathbf{E}_1(x,z) = \mathbf{E}_i(x,z) + \mathbf{E}_r(x,z) = -2j\mathbf{a}_y E_{0i} \sin(\beta_1 z \cos\theta_i)\, e^{-j\beta_1 x \sin\theta_i}$$

$$\mathbf{H}_1(x,z) = -\frac{2E_{0i}}{\eta_1}\left[j \sin\theta_i \sin(\beta_1 z \cos\theta_i)\, \mathbf{a}_z + \cos\theta_i \cos(\beta_1 z \cos\theta_i)\, \mathbf{a}_x\right] \times e^{-j\beta_1 x \sin\theta_i}$$

$$\mathbf{P}_{\text{average}} = \frac{1}{2}\Re\{\mathbf{E}_1 \times \mathbf{H}_1^*\} = \frac{2|E_{0i}|^2}{\eta_1} \sin\theta_i \sin^2(\beta_1 z \cos\theta_i)\, \mathbf{a}_x$$

Perpendicular Polarization

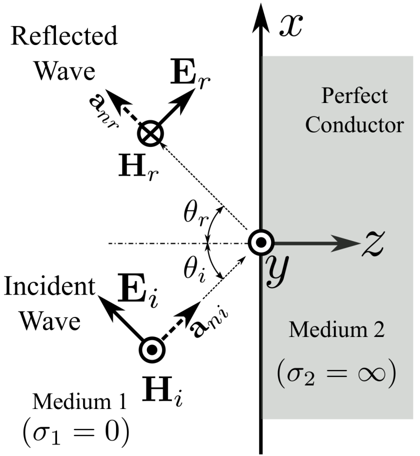
Example

A uniform plane wave ($\mathbf{E}_i$, $\mathbf{H}_i$) of an angular frequency ω is incident from air on a very large, perfectly conducting wall at an angle of incidence θ_i with perpendicular polarization. Find

1. the instantaneous surface current induced on the conducting wall, and
2. the time average Poynting vector in medium 1.

Outline

Parallel Polarization



$$\mathbf{E}_i(x,z) = E_{0i}(\cos\theta_i \mathbf{a}_x - \sin\theta_i \mathbf{a}_z) \times e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{E}_r(x,z) = E_{0r}(\cos\theta_r \mathbf{a}_x + \sin\theta_r \mathbf{a}_z) \times e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\mathbf{H}_i(x,z) = \mathbf{a}_y \frac{E_{0i}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

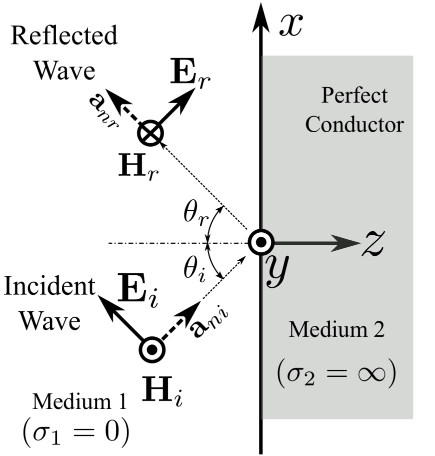
$$\mathbf{H}_r(x,z) = -\mathbf{a}_y \frac{E_{0r}}{\eta_1} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)},$$

Boundary condition: tangential electric field vanishes at $z = 0$,

$$E_{0i}\cos\theta_i e^{-j\beta_1 x\sin\theta_i} + E_{or}\cos\theta_r e^{-j\beta_1 x\sin\theta_r} = 0$$

$$\boxed{\theta_r = \theta_i, \qquad E_{0r} = -E_{0i}}$$

Parallel Polarization



$$\mathbf{E}_i(x,z) = E_{0i}(\cos\theta_i \mathbf{a}_x - \sin\theta_i \mathbf{a}_z) \times e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{E}_r(x,z) = -E_{0i}(\cos\theta_i \mathbf{a}_x + \sin\theta_i \mathbf{a}_z) \times e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\mathbf{H}_i(x,z) = \mathbf{a}_y \frac{E_{0i}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)},$$

$$\mathbf{H}_r(x,z) = \mathbf{a}_y \frac{E_{0i}}{\eta_1} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)},$$

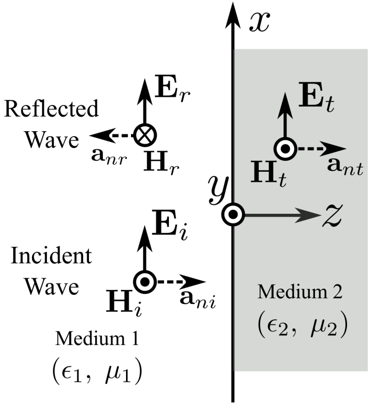
$$\mathbf{E}_1(x,z) = \mathbf{E}_i(x,z) + \mathbf{E}_r(x,z)$$

$$= -2E_{0i}\left[j\cos\theta_i \sin(\beta_1 z\cos\theta_i)\mathbf{a}_x + \sin\theta_i\cos(\beta_1 z\cos\theta_i)\mathbf{a}_z\right] e^{-j\beta_1 x\sin\theta_i}$$

$$\mathbf{H}_1(x,z) = \mathbf{H}_i(x,z) + \mathbf{H}_r(x,y) = \mathbf{a}_y \frac{2E_{0i}}{\eta_1}\cos(\beta_1 z\cos\theta_i)\, e^{-j\beta_1 x\sin\theta_i}$$

Outline

Normal Incidence at a Plane Dielectric Boundary



- Incident wave

$$\mathbf{E}_i = \mathbf{a}_x E_{0i} e^{-j\beta_1 z}, \quad \mathbf{H}_i = \mathbf{a}_y \frac{E_{0i}}{\eta_1} e^{-j\beta_1 z}$$

- Reflected wave

$$\mathbf{E}_r = \mathbf{a}_x E_{0r} e^{j\beta_1 z}, \quad \mathbf{H}_r = -\mathbf{a}_y \frac{E_{0r}}{\eta_1} e^{j\beta_1 z}$$

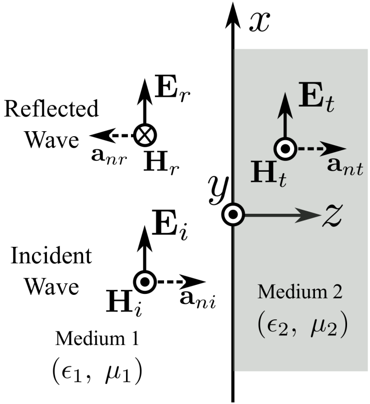
- Transmitted wave,

$$\mathbf{E}_t = \mathbf{a}_x E_{0t} e^{-j\beta_2 z}, \quad \mathbf{H}_i = \mathbf{a}_y \frac{E_{0t}}{\eta_2} e^{-j\beta_2 z}$$

- Boundary conditions: tangential electric and magnetic fields have to be continuous on the boundary at $z = 0$.

$$E_{0i} + E_{0r} = E_{0t}, \qquad \frac{E_{0i}}{\eta_1} - \frac{E_{0r}}{\eta_1} = \frac{E_{0t}}{\eta_2}$$

Normal Incidence at a Plane Dielectric Boundary



- Boundary conditions: tangential electric and magnetic fields have to be continuous on the boundary at $z = 0$.

$$E_{0i} + E_{0r} = E_{0t}, \qquad \frac{E_{0i}}{\eta_1} - \frac{E_{0r}}{\eta_1} = \frac{E_{0t}}{\eta_2}$$

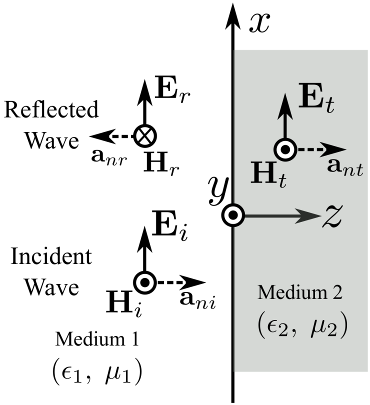
Define the reflection coefficient $\Gamma = E_{0r}/E_{0i}$ and transmission coefficient $\tau = E_{0t}/E_{0i}$.

$$\Gamma = \frac{E_{0r}}{E_{0i}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

$$\tau = \frac{E_{0t}}{E_{0i}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$\boxed{1 + \Gamma = \tau}$$

Normal Incidence at a Plane Dielectric Boundary



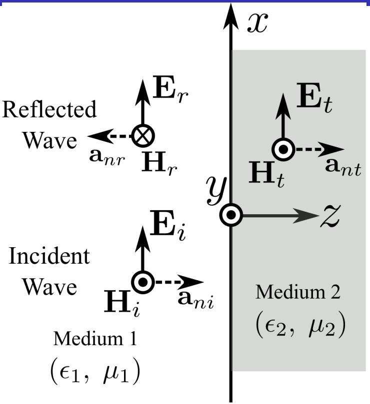
Total field in medium 1

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r$$

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{a}_x E_{i0} \left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z} \right) \\ &= \mathbf{a}_x E_{i0} \left[(1+\Gamma)\, e^{-j\beta_1 z} + \Gamma \left(e^{j\beta_1 z} - e^{-j\beta_1 z} \right) \right] \\ &= \mathbf{a}_x E_{i0} \left[\tau e^{-j\beta_1 z} + 2j\Gamma \sin\left(\beta_1 z\right) \right]\end{aligned}$$

Field is composed of a traveling wave with amplitude τE_{0i}, and a standing wave with amplitude $2\Gamma E_{0i}$.

Normal Incidence at a Plane Dielectric Boundary



$$\mathbf{E}_1 = \mathbf{a}_x E_{0i} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right)$$

- $\Gamma > 0\ (\eta_2 > \eta_1)$
  Maximum of $|\mathbf{E}_1(z)|$ is $E_{0i}(1+\Gamma)$ at
  $z_{max} = -\frac{n\lambda_1}{2},\ n = 0,\ 1,\ 2, \cdots$
  Minimum of $|\mathbf{E}_1(z)|$ is $E_{0i}(1-\Gamma)$ at
  $z_{min} = -\frac{(2n+1)\lambda_1}{4},\ n = 0,\ 1,\ 2, \cdots$

- $\Gamma < 0\ (\eta_2 < \eta_1)$
  Maximum of $|\mathbf{E}_1(z)|$ is $E_{0i}(1-\Gamma)$ at $z_{max} = -(2n+1)\lambda_1/4$, $n = 0,\ 1, \cdots$
  Minimum of $|\mathbf{E}_1(z)|$ is $E_{0i}(1+\Gamma)$ at $z_{min} = -n\lambda_1/2,\ n = 0,\ 1,\ 2, \cdots$

Standing wave ratio (SWR), $S = \frac{|\mathbf{E}_1|_{max}}{|\mathbf{E}_1|_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|}$.

Normal Incidence at a Plane Dielectric Boundary

Instantaneous electric field $E_x(z,t)$ and the amplitude envelope of the oscillating field for $\Gamma > 0$ $(\eta_2 > \eta_1)$

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Normal Incidence at a Plane Dielectric Boundary

Instantaneous electric field $E_x(z,t)$ and the amplitude envelope of the oscillating field for $\Gamma < 0$ $(\eta_2 < \eta_1)$

Electric Field

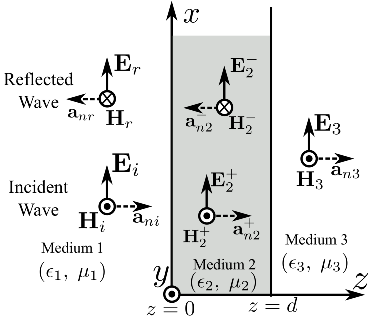
Normal Incidence at a Plane Dielectric Boundary

Example

A uniform plane wave in a lossless medium with intrinsic impedance η_1 is incident normal onto another lossless medium with intrinsic impedance η_2 through a plane boundary. Obtain the expression for the time average power densities in both media.

Outline

Normal Incidence at Multiple Dielectric Interfaces



- Medium 1 $(z < 0)$

$$\mathbf{E}_1(z) = \mathbf{a}_x \left(E_{0i} e^{-j\beta_1 z} + E_{0r} e^{j\beta_1 z} \right)$$

$$\mathbf{H}_1(z) = \mathbf{a}_y \frac{1}{\eta_1} \left(E_{0i} e^{-j\beta_1 z} - E_{0r} e^{j\beta_1 z} \right)$$

- Medium 2 $(0 < z < d)$

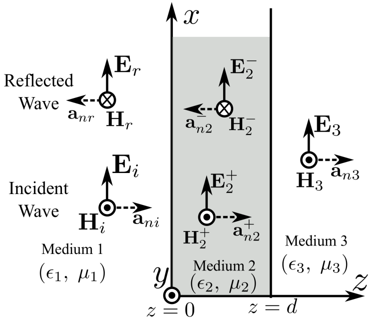
$$\mathbf{E}_2(z) = \mathbf{a}_x \left(E_2^+ e^{-j\beta_2 z} + E_2^- e^{j\beta_2 z} \right)$$

$$\mathbf{H}_2(z) = \mathbf{a}_y \frac{1}{\eta_2} \left(E_2^+ e^{-j\beta_2 z} - E_2^- e^{j\beta_2 z} \right)$$

- Medium 3 $(z > d)$

$$\mathbf{E}_3(z) = \mathbf{a}_x E_3^+ e^{-j\beta_3 z}, \qquad \mathbf{H}_2(z) = \mathbf{a}_y \frac{1}{\eta_2} E_3^+ e^{-j\beta_3 z}$$

Normal Incidence at Multiple Dielectric Interfaces



- Boundary conditions
  - at $z = 0$,
    $\mathbf{E}_1(0) = \mathbf{E}_2(0)\,, \qquad \mathbf{H}_1(0) = \mathbf{H}_2(0)$
  - at $z = d$,
    $\mathbf{E}_2(d) = \mathbf{E}_3(d)\,, \qquad \mathbf{H}_2(d) = \mathbf{H}_3(d)$

These four equations can be solved for the ratios $\frac{E_{0r}}{E_{0i}}$, $\frac{E_2^-}{E_{0i}}$, $\frac{E_2^+}{E_{0i}}$, and $\frac{E_3}{E_{0i}}$.

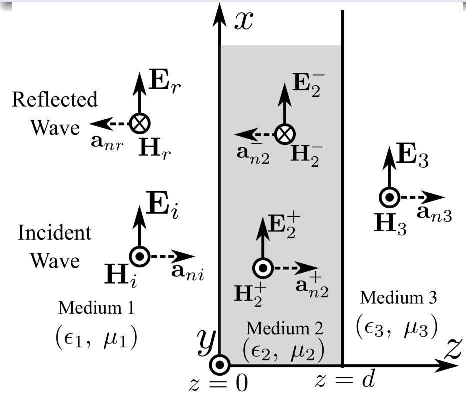
- An easier way to solve this problem using impedance of the total field.

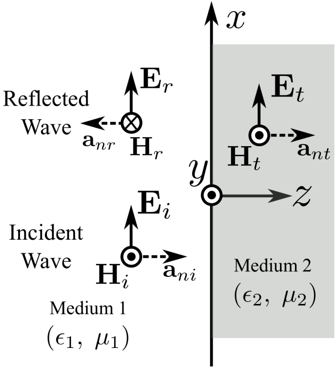
Outline

Wave Impedance of the Total Field

Definitions

The *wave impedance of the of the total field* at any plane *parallel* to the plane boundary is the ratio of the total *tangential* electric field intensity to the the total *tangential* magnetic field intensity.



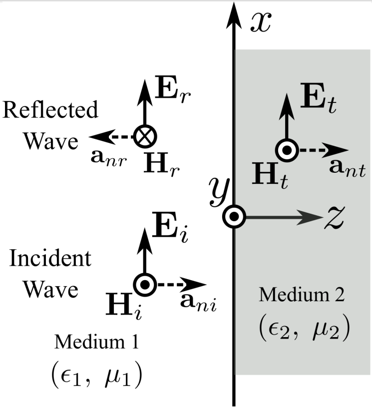


$$Z(z) = \frac{\text{Total } E_x(z)}{\text{Total } H_y(z)}$$

Wave Impedance of the Total Field

Definitions

The *wave impedance of the of the total field* at any plane *parallel* to the plane boundary is the ratio of the total *tangential* electric field intensity to the the total *tangential* magnetic field intensity.



$$E_{1x}(z) = E_{i0}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right)$$

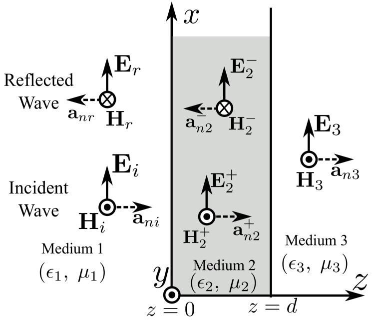
$$H_{1y}(z) = \frac{E_{i0}}{\eta_1}\left(e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}\right)$$

$$Z_1(z) = \eta_1 \frac{e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}}{e^{-j\beta_1 z} - \Gamma e^{j\beta_1 z}}$$

where $\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$.

$$\boxed{Z_1(-\ell) = \eta_1 \frac{\eta_2 \cos\beta_1\ell + j\eta_1 \sin\beta_1\ell}{\eta_1 \cos\beta_1\ell + j\eta_2 \sin\beta_1\ell}}$$

Normal Incidence at Multiple Dielectric Interfaces



$$Z_2(0) = \eta_2 \frac{\eta_3 \cos \beta_2 d + j\eta_2 \sin \beta_2 d}{\eta_2 \cos \beta_2 d + j\eta_3 \sin \beta_2 d}$$

The reflection coefficient at $z = 0$, Γ_0,

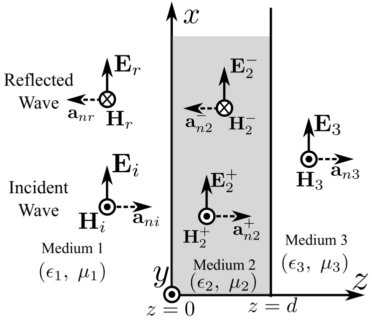
$$\Gamma_0 = \frac{E_{r0}}{E_{i0}} = -\frac{H_{r0}}{H_{i0}} = \frac{Z_2(0) - \eta_1}{Z_2(0) + \eta_1}$$

Example

A dielectric layer of thickness d and intrinsic impedance η_2 is placed between media 1 and 3 having intrinsic impedances η_1 and η_3, respectively. Determine d and η_2 such that no reflection occurs when a uniform plane wave in medium 1 impinges normally on the interface with medium 2, when

1. $\eta_3 = \eta_1$
2. $\eta_3 \neq \eta_1$

Normal Incidence at Multiple Dielectric Interfaces



$$Z_2(0) = \eta_2 \frac{\eta_3 \cos\beta_2 d + j\eta_2 \sin\beta_2 d}{\eta_2 \cos\beta_2 d + j\eta_3 \sin\beta_2 d}$$

The reflection coefficient at $z=0$, Γ_0,

$$\Gamma_0 = \frac{E_{r0}}{E_{i0}} = -\frac{H_{r0}}{H_{i0}} = \frac{Z_2(0)-\eta_1}{Z_2(0)+\eta_1}$$

Example

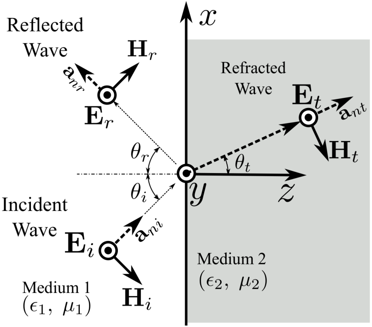
A dielectric layer of thickness d and intrinsic impedance η_2 is placed between media 1 and 3 having intrinsic impedances η_1 and η_3, respectively. Determine d and η_2 such that no reflection occurs when a uniform plane wave in medium 1 impinges normally on the interface with medium 2, when

1. $\eta_3 = \eta_1$
2. $\eta_3 \neq \eta_1$

Outline

Outline

Perpendicular Polarization



- medium 1 $(z < 0)$

$$\mathbf{E}_1 = \mathbf{a}_y \left[E_{0i} e^{-j\beta_1 (z\cos\theta_i + x\sin\theta_i)} + E_{0r} e^{-j\beta_1(-z\cos\theta_r + x\sin\theta_r)} \right]$$

- medium 2 $(z > 0)$

$$\mathbf{E}_2 = \mathbf{a}_y E_{0t} e^{-j\beta_2(z\cos\theta_t + x\sin\theta_t)}$$

- Continuity of the tangential electric field across the interface $z = 0$,

$$E_{0i} e^{-j\beta_1 x \sin\theta_i} + E_{0r} e^{-j\beta_1 x\sin\theta_r} = E_{0t} e^{-j\beta_2 x \sin\theta_t}$$

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

$$\boxed{\theta_r = \theta_i, \qquad \text{Snell's law of reflection}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t, \qquad \text{Snell's law of refraction,}}$$

where $\quad n_1 = \sqrt{\mu_{r1}\epsilon_{r1}}, \qquad n_2 = \sqrt{\mu_{r2}\epsilon_{r2}}$

$$E_{0i} + E_{0r} = E_{0t}$$

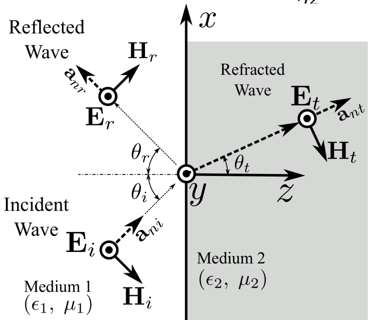
Perpendicular Polarization

- Magnetic field **H**
  - Medium 1 ($z < 0$),

$$\mathbf{H}_1 = \frac{1}{\eta_1}\Big[E_{0i}\left(-\cos\theta_i \mathbf{a}_x + \sin\theta_i \mathbf{a}_z\right) e^{-j\beta_1(z\cos\theta_i + x\sin\theta_i)} + E_{0r}\left(\cos\theta_i \mathbf{a}_x + \sin\theta_i \mathbf{a}_z\right) e^{-j\beta_1(-z\cos\theta_i + x\sin\theta_i)}\Big]$$

  - Medium 2 ($z > 0$),

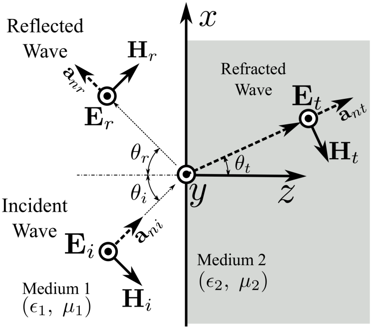
$$\mathbf{H}_2 = \frac{1}{\eta_2} E_{0t}\left(-\cos\theta_t \mathbf{a}_x + \sin\theta_t \mathbf{a}_z\right) e^{-j\beta_2(z\cos\theta_t + x\sin\theta_t)}$$



- Continuity of the tangential electric field across the interface $z = 0$,

$$\frac{\cos\theta_t}{\eta_1}\left(-E_{0i} + E_{0r}\right) = -\frac{\cos\theta_t}{\eta_2} E_{0t}$$

Perpendicular Polarization



$$E_{0i} + E_{0r} = E_{0t}$$

$$\frac{\cos\theta_i}{\eta_1}(-E_{0i} + E_{0r}) = -\frac{\cos\theta_t}{\eta_2}E_{0t}$$

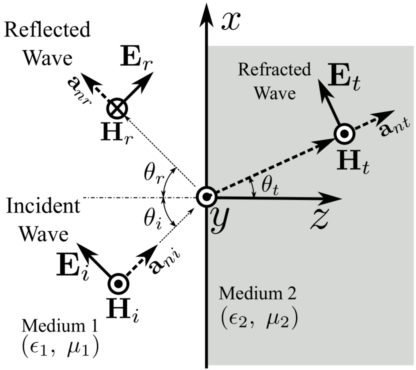
Solving for $\Gamma_\perp = E_{0r}/E_{0i}$, and $\tau_\perp = E_{0t}/E_{0i}$,

$$\Gamma_\perp = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}$$

$$\tau_\perp = \frac{2\eta_2/\cos\theta_t}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}$$

Outline

Parallel Polarization



- Magnetic field **H**
  Medium 1 ($z > 0$),

$$\mathbf{H}_1 = \frac{1}{\eta_1}\mathbf{a}_y \left[E_{0i} e^{-j\beta_1 (z\cos\theta_i + x\sin\theta_i)} - E_{0r} e^{-j\beta_1(-z\cos\theta_r + x\sin\theta_r)} \right]$$

  Medium 2 ($z < 0$),

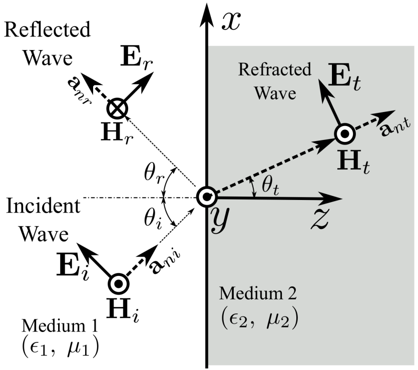
$$\mathbf{H}_2 = \frac{1}{\eta_2}\mathbf{a}_y \left[E_{0t} e^{-j\beta_2 (z\cos\theta_t + x\sin\theta_t)} \right]$$

- Continuity of the tangential magnetic field across the interface $z = 0$,

$$\left(E_{0i} e^{-j\beta_1 x\sin\theta_i} - E_{0r} e^{-j\beta_1 x\sin\theta_r} \right)/\eta_1 = E_{0t} e^{-j\beta_2 x \sin\theta_t}/\eta_2$$

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

Parallel Polarization



$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

$$\boxed{\theta_r = \theta_i, \qquad \text{Snell's law of reflection}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t, \qquad \textit{Snell's law of refraction,}}$$

$$\textit{where} \quad n_1 = \sqrt{\mu_{r1}\epsilon_{r1}}, \qquad n_2 = \sqrt{\mu_{r2}\epsilon_{r2}}$$

$$\left(E_{0i} - E_{0r}\right)/\eta_1 = E_{0t}/\eta_2$$

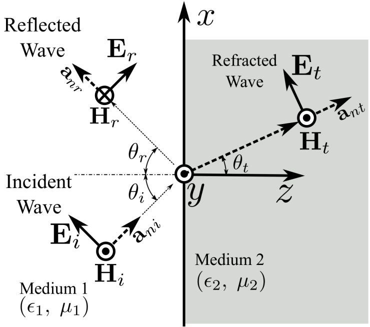
Parallel Polarization

- Magnetic field **E**
  - Medium 1 ($z < 0$),

$$\mathbf{E}_1 = E_{0i}\left(\cos\theta_i \mathbf{a}_x - \sin\theta_i \mathbf{a}_z\right) e^{-j\beta_1(z\cos\theta_i + x\sin\theta_i)}$$
$$\left. + E_{0r}\left(\cos\theta_i \mathbf{a}_x + \sin\theta_i \mathbf{a}_z\right) e^{-j\beta_1(-z\cos\theta_i + x\sin\theta_i)}\right]$$

  - Medium 2 ($z > 0$),

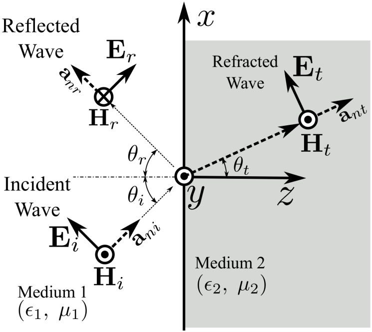
$$\mathbf{E}_2 = E_{0t}\left(\cos\theta_t \mathbf{a}_x - \sin\theta_t \mathbf{a}_z\right) e^{-j\beta_2(z\cos\theta_t + x\sin\theta_t)}$$



- Continuity of the tangential electric field across the interface $z = 0$,

$$\cos\theta_i\left(E_{0i} + E_{0r}\right) = \cos\theta_t E_{0t}$$

Parallel Polarization



$$(E_{0i} - E_{0r})/\eta_1 = E_{0t}/\eta_2$$

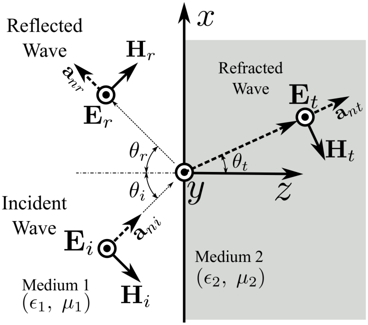
$$\cos\theta_i (E_{0i} + E_{0r}) = \cos\theta_t E_{0t}$$

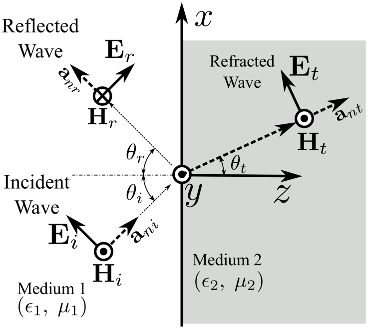
Solving for $\Gamma_\parallel = E_{0r}/E_{0i}$, and $\tau_\parallel = E_{0t}/E_{0i}$,

$$\Gamma_\parallel = \frac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

$$\tau_\parallel = \frac{2\eta_2\cos\theta_t}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$$

Solving oblique incidence using wave impedance





$$Z = \frac{E_y}{H_x} = \begin{cases} Z_1 = \dfrac{\eta_1}{\cos\theta_i} & \text{med. } 1 \\ Z_2 = \dfrac{\eta_2}{\cos\theta_t} & \text{med. } 2 \end{cases} \qquad Z = \frac{E_x}{H_y} = \begin{cases} Z_1 = \eta_1 \cos\theta_i & \text{med. } 1 \\ Z_2 = \eta_2 \cos\theta_t & \text{med. } 2 \end{cases}$$

$$\Gamma = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

The impedance at $z = -d$ is given by,

$$Z_1(-d) = Z_1 \frac{Z_2 \cos\beta_{1z} d + jZ_1 \sin\beta_{1z} d}{Z_1 \cos\beta_{1z} d + jZ_2 \sin\beta_{1z} d},$$

Oblique Incidence

Snell's laws are applied for both perpendicular and parallel polarization.

$$\boxed{\theta_r = \theta_i, \qquad \text{Snell's law of reflection}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t, \qquad \textit{Snell's law of refraction},}$$

$$\textit{where} \quad n_1 = \sqrt{\mu_{r1}\epsilon_{r1}}, \qquad n_2 = \sqrt{\mu_{r2}\epsilon_{r2}}$$

Outline

Brewster Angle

Definition

Brewster angle θ_B is angle of incident where there is no reflection $\Gamma = 0$.

- Perpendicular polarization,

$$\Gamma_\perp = \frac{\eta_2 / \cos\theta_t - \eta_1 / \cos\theta_i}{\eta_2 / \cos\theta_t + \eta_1 / \cos\theta_i} = 0 \quad \Longrightarrow \quad \eta_2 \cos\theta_{B\perp} = \eta_1 \cos\theta_t$$

Snell's law of refraction: $\quad n_1 \sin\theta_{B\perp} = n_2 \sin\theta_t$

$$\sin^2\theta_{B\perp} = \frac{1 - (\eta_1/\eta_2)^2}{1 - \left(\dfrac{n_1\eta_1}{n_2\eta_2}\right)^2} = \frac{1 - \dfrac{\mu_1\epsilon_2}{\mu_2\epsilon_1}}{1 - (\mu_1/u_2)^2}$$

$\theta_{B\perp}$ does not exist for nonmagnetic material, $\mu_1 = \mu_2$.

Brewster Angle

- Parallel polarization,

$$\Gamma_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_{B\parallel}}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_{B\parallel}} = 0 \quad \Longrightarrow \quad \eta_1 \cos\theta_{B\parallel} = \eta_2 \cos\theta_t$$

Snell's law of refraction: $\quad n_1 \sin\theta_{B\parallel} = n_2 \sin\theta_t$

$$\sin^2\theta_{B\parallel} = \frac{1 - (\eta_2/\eta_1)^2}{1 - \left(\dfrac{n_1\eta_2}{n_2\eta_1}\right)^2} = \frac{1 - \dfrac{\mu_2\epsilon_1}{\mu_1\epsilon_2}}{1 - (\epsilon_1/\epsilon_2)^2}$$

For nonmagnetic material $\mu_1 = \mu_2$,

$$\sin\theta_{B\parallel} = \frac{1}{\sqrt{1 + \epsilon_1/\epsilon_2}}$$

Brewster Angle

Example

The dielectric constant of pure water is 80.

1. Determine the Brewster angle for parallel polarization $\theta_{B\|}$, and the corresponding angle of transmission.
2. A plane wave with perpendicular polarization is incident from air on water surface at $\theta_i = \theta_{B\|}$. Find the reflection and transmission coefficients.

Outline

Oblique Incidence

- Snell's laws are applied for both perpendicular and parallel polarization.

$$\boxed{\theta_r = \theta_i, \qquad \text{Snell's law of reflection}}$$

$$\boxed{n_1 \sin\theta_i = n_2 \sin\theta_t, \qquad \textit{Snell's law of refraction},}$$
$$\textit{where} \quad n_1 = \sqrt{\mu_{r1}\epsilon_{r1}}, \qquad n_2 = \sqrt{\mu_{r2}\epsilon_{r2}}$$

Definition

For media with $n_1 > n_2$ the critical angle θ_c is the angle of incidence for which the transmitted angle $\theta_t = \pi/2$.

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

For nonmagnetic material $\mu_1 = \mu_2$,

$$\theta_c = \sin^{-1}\left(\sqrt{\frac{\epsilon_2}{\epsilon_1}}\right)$$

Total Internal Reflection

If $\theta_i > \theta_c$, Snell's law of refraction gives,

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i > 1, \quad \Longrightarrow \quad \cos\theta_t = \pm j\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}$$

Hence the propagation in medium 2,

$$e^{-j\beta_{2x}x}e^{\mp\alpha_2 z}$$

where the positive sign is not taken as it gives a growing field with z.

$$e^{-j\beta_{2x}x}e^{-\alpha_2 z}$$

$$\beta_{2x} = \beta_2\sin\theta_t = \beta_2\frac{n_1}{n_2}\sin\theta_i$$

$$\alpha_2 = \beta_2\sqrt{\frac{n_1^2}{n_2^2}\sin^2\theta_i - 1}$$

For total internal reflection

$$|\Gamma| = 1$$

Example

Example

The permittivity of water at optical frequencies is $1.75\epsilon_0$. It is found that that an isotropic light source at a distance d under water yields an illuminated circular area of radius 5 m. Determine d.

Example

A dielectric rod or fiber of a transparent material can be used to guide light or an electromagnetic wave under the condition of total internal reflection. Determine the minimum dielectric constant of the guiding medium so that a wave incident on one end at any angle will be confined within the rod until it emerges from the other end.