Electromagnetic Waves

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Electromagnetic Waves

Outline

Source-Free Wave Equations in Homogeneous Medium

1. $\nabla \cdot \mathscr{E} = 0,$
2. $\nabla \cdot \mathscr{H} = 0$
3. $\nabla \times \mathscr{E} = -\mu \dfrac{\partial \mathscr{H}}{\partial t},$
4. $\nabla \times \mathscr{H} = \varepsilon \dfrac{\partial \mathscr{E}}{\partial t},$

Those equations give the wave equations for the electric and magnetic fields,

$$\nabla^2 \mathscr{E} - \frac{1}{u^2} \frac{\partial^2 \mathscr{E}}{\partial t^2} = \mathbf{0}$$

$$\nabla^2 \mathscr{H} - \frac{1}{u^2} \frac{\partial^2 \mathscr{H}}{\partial t^2} = \mathbf{0}$$

Outline

Time-Harmonic Electromagnetics

Instantaneous fields of a time harmonic dependence can be written as,

$$\mathscr{E}(x,y,z,t) = \Re\left\{\mathbf{E}(x,y,z)\,e^{j\omega t}\right\}$$

Other fields and sources quantities can be written similarly.

1. $\nabla\cdot\mathbf{D} = \rho_f,$ (Gauss law),
2. $\nabla\cdot\mathbf{B} = 0$
3. $\nabla\times\mathbf{E} = -j\omega\mathbf{B},$ (Faraday's Law)
4. $\nabla\times\mathbf{H} = \mathbf{J}_f + j\omega\mathbf{D},$ (Modified Ampere's Law)

where

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

For linear isotropic materials,

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m\mathbf{H}$$

$$\mathbf{D} = \varepsilon\mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

where $\varepsilon = \varepsilon_0(1+\chi_e)$, and $\mu = \mu_0(1+\chi_m)$.

Potential Functions for Time-Harmonic Fields

$$\boxed{\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}}$$

Gives the solution,

$$\boxed{\mathbf{A} = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J} e^{-jkR}}{R} dv'}$$

where $u = 1/\sqrt{\mu\varepsilon}$ is the velocity of light.
Similarly,

$$\boxed{\nabla^2 V + k^2 V = -\rho/\varepsilon}$$

Gives the solution,

$$\boxed{V(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon} \int_{V'} \frac{\rho e^{-jkR}}{R} dv'}$$

Outline

Source-Free Fields in Simple Media

1. $\nabla \cdot \mathbf{E} = 0,$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H},$
4. $\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E},$

The electrics and magnetic fields satisfy the homogeneous Helmholtz equations,

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

$$\nabla^2\mathbf{H} + k^2\mathbf{H} = 0$$

Example

Show that if $(\mathbf{E}, \mathbf{H})$ are solutions of the source-free Maxwell's equations in a simple medium characterized with ε and μ, then so also are $(\mathbf{E}', \mathbf{H}')$, where $\mathbf{E}' = \eta\mathbf{H}$, and $\mathbf{H}' = -\mathbf{E}/\eta$, and the intrinsic impedance $\eta = \sqrt{\mu/\varepsilon}$.

Outline

Plane Wave in Lossless Media

In free space, source free, the Helmholtz equation,

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}+k_0^2\right)\mathbf{E}=\mathbf{0}$$

For 1 dimensional problem, i.e. dependence on z, we can write,

$$\frac{\partial^2\mathbf{E}}{\partial z^2}+k_0^2\mathbf{E}=\mathbf{0}$$

Consider $\mathbf{E}=E_x(z)\,\mathbf{a}_x$,

$$\frac{\partial^2 E_x}{\partial z^2}+k_0^2 E_x=0$$

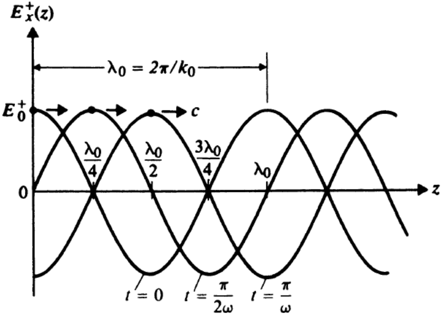
$$E_x(z)=E_0^+e^{-jkz}+E_0^-e^{jkz}$$

Examining only the solution, with $E_0^-=0$ and E_0^+real,

$$E_x(z,t)=\Re\left\{E_0^+e^{-jkz}e^{j\omega t}\right\}=E_0^+\cos(\omega t-kz)$$

Plane Wave in Lossless Media

$$\mathscr{E}_x(z,t) = \Re\left\{E_0^+ e^{-jkz} e^{j\omega t}\right\} = E_0^+ \cos(\omega t - kz)$$



- Free space wavenumber k_0,

$$k_0 = \frac{2\pi}{\lambda_0}$$

  $\lambda_0$is the free space wavelength.

- Magnetic field $\mathscr{H}$,

$$\mathscr{H}(z,t) = \mathbf{a}_y \frac{E_0^+}{\eta_0} \cos(\omega t - k_0 z)$$

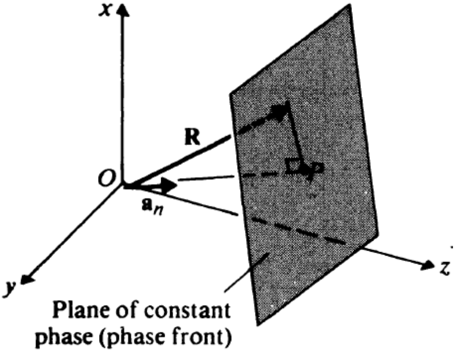
$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi\,\Omega$, is the intrinsic impedance of free space.

Transverse Electromagnetic Waves

$$\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z}$$

Helmholtz equation gives,

$$k_x^2 + k_y^2 + k_z^2 = k_0^2 = \omega^2 \mu \varepsilon$$



- Wavenumber vector $\mathbf{k}$,

$$\mathbf{k} = k_x \mathbf{a}_x + k_y \mathbf{a}_y + k_z \mathbf{a}_z = k \mathbf{a}_n$$

The radius vector $\mathbf{R}$,

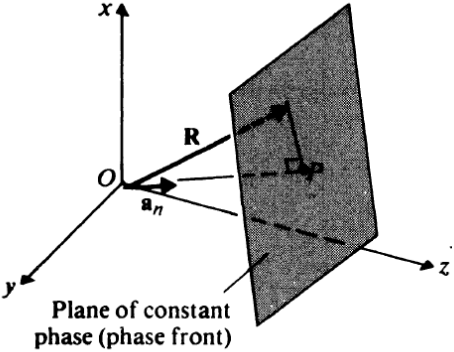
$$\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$$

$$\mathbf{E}(\mathbf{R}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}}$$

$\mathbf{a}_n \cdot \mathbf{R} = \text{Length } \overline{OP}$ (a constant)

Transverse Electromagnetic Waves

$$\mathbf{E}(x, y, z) = \mathbf{E}_0 e^{-jk_x x - jk_y y - jk_z z}$$



- $\nabla \cdot \mathbf{E} = 0$, gives,

$$\mathbf{k} \cdot \mathbf{E}_0 = 0 \longrightarrow \mathbf{E} \perp \mathbf{a}_n$$

- $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}, \quad \Longrightarrow$
  $-j\mathbf{k} \times \mathbf{E} = -j\omega\mu\mathbf{H},$

$$\mathbf{H} = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E} = \frac{1}{\eta}\mathbf{a}_n \times \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{R}}$$

$$\mathbf{H} = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{R}}$$

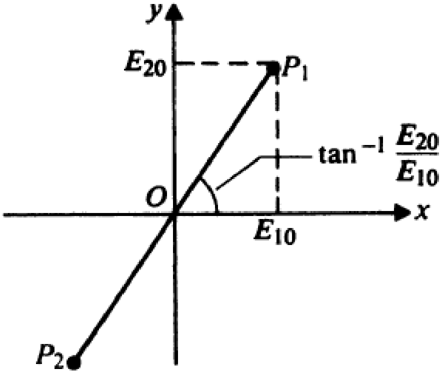
Polarization of Plane Waves

Definition

Polarization of a wave is the direction of the electric field **E**.

$$\mathbf{E}(z) = (E_{01}\mathbf{a}_x + E_{02}\mathbf{a}_y)\, e^{-jkz}$$

- If E_{01} and E_{02} are in phase, we have linear polarization.



Polarization of Plane Waves

$$\mathbf{E}(z) = (E_{10}\mathbf{a}_x + E_{20}\mathbf{a}_y)\, e^{-jkz}$$

- If $|E_{10}| = |E_{20}| = E_0$ and phase shift is 90°, we have circular polarization:
    - $\mathbf{E}(z) = E_0(\mathbf{a}_x - j\mathbf{a}_y)\, e^{-jkz}$, Right-Hand, or positive polarized wave.
    - $\mathbf{E}(z) = E_0(\mathbf{a}_x + j\mathbf{a}_y)\, e^{-jkz}$, Left-Hand, or negative polarized wave.

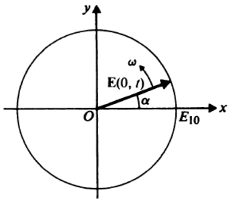


Figure: Right-Hand or Positive Circularly Polarized Wave

- Otherwise the wave is elliptically polarized.

Sense of polarization

Rotation of the field is from the leading to the lagging component.

Outline

Phase Velocity

The instantaneous field of any of the components can be written as,

$$E_x(z,t) = \Re\left\{E_0^+ e^{-jkz} e^{j\omega t}\right\} = E_0^+ \cos(\omega t - kz + \phi_0) = E_0^+ \cos(\phi(z,t)),$$

where $\phi(z,t) = \omega t - kz + \phi_0$.

Definition

Phase velocity is the speed of a point moving such that it sees a constant phase of the wave, i.e.

$$d\phi(z,t) = \omega dt - k dz = 0, \qquad \longrightarrow \qquad \frac{dz(t)}{dt} = \frac{\omega}{k}$$

$$\boxed{v_p = \frac{\omega}{k}}$$

Outline

Conductive Medium

Simple conducting medium will have $\mathbf{J} = \sigma\mathbf{E}$. Therefore,

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E} + \mathbf{J} = j\omega\left(\varepsilon + \frac{\sigma}{j\omega}\right)\mathbf{E}$$

Define complex permittivity ,

$$\boxed{\varepsilon_c = \varepsilon + \frac{\sigma}{j\omega} = \varepsilon - j\frac{\sigma}{\omega} = \varepsilon' - j\varepsilon''}$$

$$\varepsilon' = \varepsilon, \qquad \varepsilon'' = \frac{\sigma}{\omega}$$

The loss tangent $\tan\delta$ is defined as,

$$\tan\delta = \frac{\varepsilon''}{\varepsilon'} = \frac{\sigma}{\omega\varepsilon}$$

- Good conductor $\dfrac{\sigma}{\omega\varepsilon} \gg 1$
- Good insulator $\dfrac{\sigma}{\omega\varepsilon} \ll 1$

Plane Waves in Conductive Media

$$\nabla^2 \mathbf{E} + k_c^2 \mathbf{E} = 0$$

$$k_c^2 = \omega^2 \mu \varepsilon_c$$

Definition

Propagation constant γ is defined by,

$$\gamma = \alpha + j\beta = jk_c = j\omega\sqrt{\mu\varepsilon_c}$$

$$\mathbf{E}(z) = \mathbf{E}_0 e^{-\gamma z} = \mathbf{E}_0 e^{-\alpha z} e^{-j\beta z},$$

$$\gamma^2 = -\omega^2 \mu \varepsilon_c$$

Low-Loss Dielectrics

- Propagation constant

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu\varepsilon'}\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}}$$

$$\gamma = \alpha + j\beta \cong j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]$$

$$\alpha = \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}}\ \text{Np/m}, \qquad \beta = \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right]\ \text{rad/m}$$

- Intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \cong \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right)$$

Good Conductors

- Propagation constant

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon_c} \cong j\omega\sqrt{\mu\varepsilon}\sqrt{-j\frac{\sigma}{\omega\varepsilon}}$$

$$\gamma = \alpha + j\beta \cong \sqrt{j}\sqrt{\omega\mu\sigma} = (1+j)\sqrt{\pi f\mu\sigma}$$

$$\alpha = \beta = \sqrt{\pi f\mu\sigma}$$

- Intrinsic impedance

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} = \sqrt{\frac{\mu}{\varepsilon}}\left(1 - j\frac{\sigma}{\omega\varepsilon}\right)^{-1/2} \cong \sqrt{\frac{\mu}{\varepsilon}}\sqrt{j\frac{\omega\varepsilon}{\sigma}} \cong (1+j)\sqrt{\frac{\pi f\mu}{\sigma}}$$

$$\eta_c = (1+j)\frac{\alpha}{\sigma}$$

Good Conductors

Definition

Skin depth or penetration depth δ,

$$\delta \equiv \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

Skin Depths, δ in (mm), of Various Materials

Material	σ (S/m)	f = 60 (Hz)	1 (MHz)	1 (GHz)
Silver	6.17×10^7	8.27 (mm)	0.064 (mm)	0.0020 (mm)
Copper	5.80×10^7	8.53	0.066	0.0021
Gold	4.10×10^7	10.14	0.079	0.0025
Aluminum	3.54×10^7	10.92	0.084	0.0027
Iron ($\mu_r \cong 10^3$)	1.00×10^7	0.65	0.005	0.00016
Seawater	4	32 (m)	0.25 (m)	†

† The ϵ of seawater is approximately $72\epsilon_0$. At $f = 1$ (GHz), $\sigma/\omega\epsilon \cong 1$ (not $\gg 1$). Under these conditions, seawater is not a good conductor, and Eq. (8–57) is no longer applicable.

Example

Example

(Cheng Example 8-4) The electric field intensity of a linearly polarized plane wave propagating in the +z-direction in seawater is $\mathscr{E} = \mathbf{a}_x 100 \cos\left(10^7 \pi t\right)$ (V/m) at $z = 0$. The constitutive parameters of seawater are $\varepsilon_r = 72$, $\mu_r = 1$, and $\sigma = 4$ (S/m).

1. Determine the attenuation constant, phase constant, intrinsic impedance, phase velocity, wavelength, and skin depth.
2. Find the distance at which the amplitude of $\mathbf{E}$ is 1% of its value at $z = 0$.
3. Write the expressions for $\mathbf{E}(z, t)$ and $\mathbf{H}(z, t)$ at $z = 0.8$ m as a function of t.

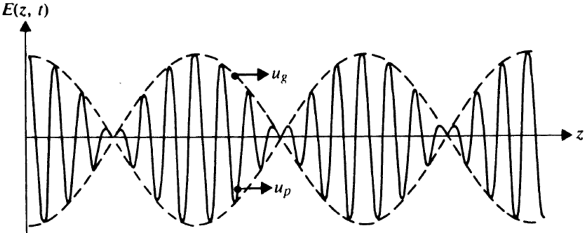
Outline

Group Velocity

For a wave packet consisting of two traveling waves of equal amplitudes and slightly shifted in frequencies at $\omega_0 - \Delta\omega$ and $\omega_0 + \Delta\omega$ $(\Delta\omega \ll \omega_0)$,

$$\mathscr{E}(z,t) = E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z]$$

$$\mathscr{E}(z,t) = 2E_0 \cos(\Delta\omega t - \Delta\beta z)\cos(\omega_0 t - \beta_0 z)$$



Velocity of the envelope is called the group velocity v_g and is obtained by,

$$\Delta\omega t - \Delta\beta z = \text{const.}$$

$$v_g \equiv \frac{d\omega}{d\beta}$$

Ionized Gases

At height 50 km - 500 km above the earth see level solar radiation causes ionization and a layer of ionized gases with equal electron and ion densities called *plasma* is formed.

$$-e\mathbf{E} = m\frac{d^2\mathbf{x}}{dt^2} = -m\omega^2\mathbf{x}, \qquad \Longrightarrow \qquad \mathbf{x} = \frac{e}{m\omega^2}\mathbf{E}$$

$$\text{dipole moment} \qquad \mathbf{p} = -e\mathbf{x} = \frac{-e^2}{m\omega^2}\mathbf{E}$$

$$\text{Polarization vector} \qquad \mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E}$$

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} = \varepsilon_0\left(1 - \frac{Ne^2}{m\omega^2\varepsilon_0}\right)\mathbf{E} = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E} = \varepsilon_p\mathbf{E}$$

ε_p is the permittivity of the plasma, and ω_p is the plasma oscillation frequency $\omega_p = \sqrt{\dfrac{Ne^2}{m\varepsilon_0}} = 2\pi f_p$, and $f_p = 9\sqrt{N}$ Hz, where N is the number density in m^{-3}.

Ionized Gases

$$\varepsilon_p = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right)$$

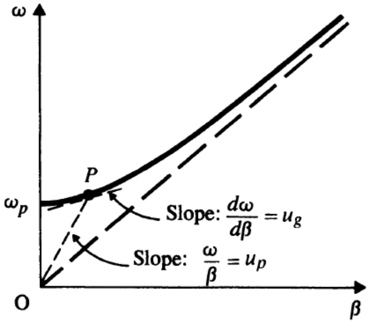
$$\beta = \omega\sqrt{\mu_0 \varepsilon_p} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

- No propagation at frequencies $\omega < \omega_p$
- Phase velocity v_p,

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} > c$$

- Group Velocity v_g,

$$v_g = \frac{d\omega}{d\beta} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c$$



Outline

- Time Dependent (Instantaneous) Maxwell's curl equations
  1. $\nabla \times \mathscr{E} = -\dfrac{\partial \mathscr{B}}{\partial t}$,
  2. $\nabla \times \mathscr{H} = \mathscr{J} + \dfrac{\partial \mathscr{D}}{\partial t}$,
- Using the vector identity

$$\nabla \cdot (\mathscr{E} \times \mathscr{H}) = (\nabla \times \mathscr{E}) \cdot \mathscr{H} - \mathscr{E} \cdot (\nabla \times \mathscr{H})$$

$$\nabla \cdot (\mathscr{E} \times \mathscr{H}) = -\frac{\partial \mathscr{B}}{\partial t} \cdot \mathscr{H} - \mathscr{E} \cdot \frac{\partial \mathscr{D}}{\partial t} - \mathscr{E} \cdot \mathscr{J}$$

$$\nabla \cdot (\mathscr{E} \times \mathscr{H}) = -\frac{\partial}{\partial t} \left(\frac{\varepsilon}{2} \mathscr{E}^2 + \frac{\mu}{2} \mathscr{H}^2 \right) - \mathscr{E} \cdot \mathscr{J}$$

$$\oint_S (\mathscr{E} \times \mathscr{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V \left(\frac{\varepsilon}{2} \mathscr{E}^2 + \frac{\mu}{2} \mathscr{H}^2 \right) dv - \int_V \mathscr{E} \cdot \mathscr{J} \, dv$$

Flow of Electromagnetic Power and The Poynting Vector

$$\oint_S (\mathscr{E} \times \mathscr{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V \left(\frac{\varepsilon}{2} \mathscr{E}^2 + \frac{\mu}{2} \mathscr{H}^2 \right) dv - \int_V \mathscr{E} \cdot \mathscr{J} \, dv$$

Definition

Poynting vector (instantaneous), $\boxed{\mathscr{P} = \mathscr{E}(\mathbf{r}, t) \times \mathscr{H}(\mathbf{r}, t)}$

Poynting Theorem (Instantaneous Form)

$$\boxed{\oint_S \mathscr{P} \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V (\mathrm{w}_e + \mathrm{w}_m) \, dv - \int_V \mathrm{p}_\sigma dv}$$

where,

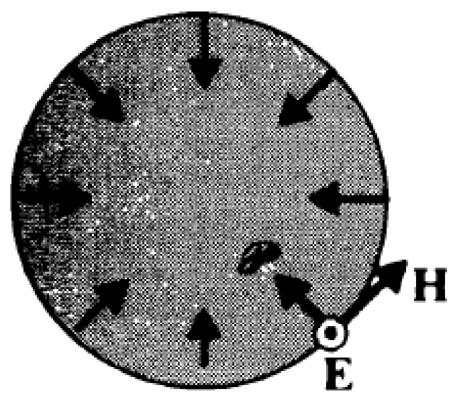
$$\mathrm{w}_e = \frac{1}{2} \varepsilon \mathscr{E}^2 = \frac{1}{2} \varepsilon \mathscr{E} \cdot \mathscr{E} = \text{Electric energy density,}$$

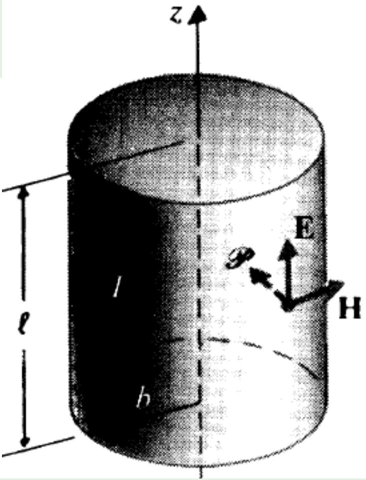
$$\mathrm{w}_m = \frac{1}{2} \mu \mathscr{H}^2 = \frac{1}{2} \mu \mathscr{H} \cdot \mathscr{H} = \text{Magnetic energy density,}$$

$$\mathrm{p}_\sigma = \sigma \mathscr{E}^2 = \frac{\mathscr{J}^2}{\sigma} = \sigma \mathscr{E} \cdot \mathscr{E} = \text{Ohmic power loss density,}$$

Example

Find the Poynting vector on the surface of a long, straight conducting wire (of radius b and conductivity σ) that carries a direct I. Verify Poynting's theorem.





Instantaneous and Average Power Densities

	Instantaneous	Average
Poynting vector	$\mathscr{P} = \boldsymbol{\mathscr{E}}(\mathbf{r}, t) \times \boldsymbol{\mathscr{H}}(\mathbf{r}, t)$	$\mathbf{P}_{av} = \frac{1}{2}\Re\left\{\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}^*(\mathbf{r}, t)\right\}$
Electric Energy Density	$\mathrm{w}_e = \frac{1}{2}\varepsilon\mathscr{E}^2 = \frac{1}{2}\varepsilon\boldsymbol{\mathscr{E}} \cdot \boldsymbol{\mathscr{E}}$	$w_{e,\,av} = \frac{1}{4}\varepsilon\left\|E\right\|^2 = \frac{1}{4}\varepsilon\mathbf{E} \cdot \mathbf{E}^*$
Magnetic Energy Density	$\mathrm{w}_m = \frac{1}{2}\mu\mathscr{H}^2 = \frac{1}{2}\mu\boldsymbol{\mathscr{H}} \cdot \boldsymbol{\mathscr{H}}$	$w_{m,\,av} = \frac{1}{4}\mu\left\|H\right\|^2 = \frac{1}{4}\mu\mathbf{H} \cdot \mathbf{H}^*$
Ohmic Power Density	$\mathrm{p}_\sigma = \sigma\mathscr{E}^2 = \frac{\mathscr{J}^2}{\sigma} = \sigma\boldsymbol{\mathscr{E}} \cdot \boldsymbol{\mathscr{E}}$	$p_{\sigma,\,av} = \frac{\sigma}{2}\left\|\mathbf{E}\right\|^2 = \frac{\left\|\mathbf{J}\right\|^2}{2\sigma} = \frac{\sigma}{2}\mathbf{E} \cdot \mathbf{E}$

Complex Poynting Theorem

- Phasor Maxwell's curl equations
  1. $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$,
  2. $\nabla \times \mathbf{H} = \mathbf{J} + j\omega\varepsilon\mathbf{E}$,
- Using the vector identity

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = (\nabla \times \mathbf{E}) \cdot \mathbf{H}^* - \mathbf{E} \cdot (\nabla \times \mathbf{H}^*)$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = -j\omega\mu\mathbf{H} \cdot \mathbf{H}^* + j\omega\varepsilon\mathbf{E} \cdot \mathbf{E}^* - \mathbf{E} \cdot \mathbf{J}^*$$

$$\frac{1}{2}\nabla \cdot (\mathbf{E} \times \mathbf{H}^*) = 2j\omega\left(\frac{\varepsilon}{4}|E|^2 - \frac{\mu}{4}|H|^2\right) - \frac{\sigma}{2}|E|^2$$

$$\boxed{\frac{1}{2}\oint_S (\mathbf{E} \times \mathbf{H}^*) \cdot d\mathbf{s} = 2j\omega\int_V (w_{e,av} - w_{m,av})\,dv - \int_V p_\sigma\,dv}$$

Example

The far field of a short vertical current element $Id\ell$ located at the origin of a spherical coordinate system in free space is

$$\mathbf{E}(R,\theta) = \mathbf{a}_\theta E_\theta(R,\theta) = \mathbf{a}_\theta j \frac{60\pi I d\ell}{\lambda R} e^{-jkR}$$

and

$$\mathbf{H}(R,\theta) = \mathbf{a}_\phi H_\phi(R,\theta) = \mathbf{a}_\phi j \frac{I d\ell}{2\lambda R} e^{-jkR},$$

where $\lambda = 2\pi/k$ is the free space wavelength.

1. Write the expression for the instantaneous Poynting vector.
2. Find the total average power radiated by the current element.