Potentials and Fields

Topic 2

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Maxwell's Equations

1. $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho, \qquad \text{(Gauss law)},$
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(Faraday's Law)}$
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \text{(Modified Ampere's Law)}$

- We are seeking solution for $\mathbf{E}(\mathbf{r}, t)$, and $\mathbf{B}(\mathbf{r}, t)$.
- This would be achieved by representing those fields in terms of potentials.

Potential Functions

- $\nabla \cdot \mathbf{B} = 0, \quad \Longrightarrow \quad \boxed{\mathbf{B} = \nabla \times \mathbf{A}}$
- $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, (Faraday's Law) $\Longrightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = \mathbf{0}, \quad \Longrightarrow \quad \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}}$$

- $\nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \varepsilon \frac{\partial \mathbf{E}}{\partial t}$, (Modified Ampere's Law) (Assuming homogeneous media)

$$\nabla \times \nabla \times \mathbf{A} = \mu \mathbf{J} + \mu \varepsilon \frac{\partial}{\partial t}\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right)$$

Using the vector identity, $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$$\nabla^2 \mathbf{A} - \mu \varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla\left(\nabla \cdot \mathbf{A} + \mu \varepsilon \frac{\partial V}{\partial t}\right)$$

Potential Functions (Continued)

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \nabla \left(\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} \right)$$

We can arbitrarily choose $\nabla \cdot \mathbf{A}$. We let,

$$\boxed{\nabla \cdot \mathbf{A} + \mu\varepsilon \frac{\partial V}{\partial t} = 0,} \qquad \text{(Lorentz Gauge)}$$

Hence,

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}}$$

- $\nabla \cdot \mathbf{E} = \rho/\varepsilon$ (Gauss's Law) (Assuming homogeneous media)

$$\nabla \cdot \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) = \rho/\varepsilon$$

Using Lorentz Gauge,

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon}$$

Retarded Potentials

The solutions to the two equations,

$$\boxed{\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\rho/\varepsilon}$$

$$\boxed{\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}}$$

are given by,

$$\boxed{V(\mathbf{r},t) = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau', \qquad \mathbf{A}(\mathbf{r},t) = \frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau'}$$

where t_r is the retarded time defined as,

$$\boxed{t_r \equiv t - \mathscr{r}/c}$$

Example

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An infinite straight wire carries the current

$$I(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ I_0 & \text{for t>0} \end{cases}$$

That is, a constant current I_0 is turned on abruptly at $t = 0$. Find the resulting electric and magnetic fields.

Hint:

$$\int \frac{dz}{\sqrt{z^2 + s^2}} = \ln\left(\sqrt{z^2 + s^2} + z\right)$$