Electric and Magnetic Fields
ELC 205B
Electrodynamics

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Electrodynamics

Course Outline

1. Electrodynamics
   - Review (Electromotive Force) and Electromagnetic Induction
   - Maxwell's Equations
2. Potential Functions
3. Electromagnetic Waves
   - Source-Free Wave Equations
   - Plane Wave Waves in Lossless and Lossy Media
   - Flow of Electromagnetic Power and The Poynting Vector
4. Plane Wave Reflection and Transmission
   - Normal and Oblique Incidence at a Plane Conducting Boundary
   - Normal and Oblique Incidence at a Plane Dielectric Boundary
   - Brewster Angle and Total Reflection

References

[1] D. J. Griffiths, Introduction to Electrodynamics, 4 edition. Boston: Addison-Wesley, 2012.
[2] D. K. Cheng, Fundamentals of Engineering Electromagnetics. Addison-Wesley Publishing Company, 1993.

Ohm's Law

For most substances, the current density $\mathbf{J}$ is proportional to the force per unit charge, $\mathbf{f}$:

$$\boxed{\mathbf{J} = \sigma \mathbf{f},} \qquad \text{Ohm's Law}$$

whereσ is called the conductivity of the medium.

- Perfect conductors, with $\sigma = \infty$.
- Insulators we can pretend $\sigma = 0$.

In principle, the force that drives the charges to produce the current could be anything-chemical, gravitational, etc..(i.e. a Lorentz force would produce, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.)

Steady Current Flow

Fact

Charge Continuity:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

For current flow under electric field $\mathbf{J} = \sigma \mathbf{E}$,

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}, \text{ and as } \quad \nabla \cdot \mathbf{E} = \rho / \varepsilon \Longrightarrow \quad -\frac{\sigma}{\varepsilon} \rho = \frac{\partial \rho}{\partial t}$$

Last equation has the solution $\rho = \rho_0 e^{-\sigma t / \varepsilon}$, and as $t \to \infty$, $\rho \to 0$.

Corollary

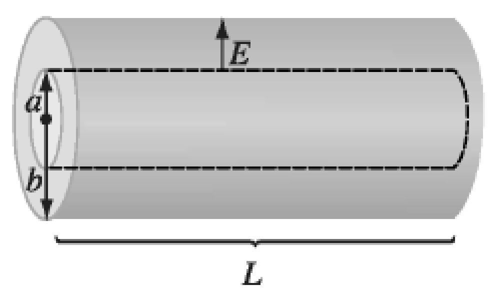
Under steady current flow in a conductor,

$$\rho = 0, \qquad \text{and} \qquad \nabla \cdot \mathbf{J} = 0$$

Ohm's Law

Example

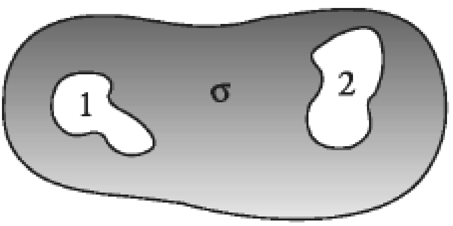
Two long coaxial metal cylinders (radii a and b) are separated by material of conductivity σ as shown in the figure. If they are maintained at a potential difference V, what current flows from one to the other, in a length L?



Example

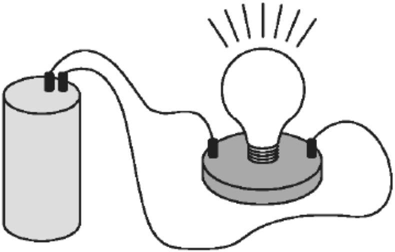
Two metal objects are embedded in weakly conducting material of conductivity σ as shown in the figure. Show that the resistance between them is related to the capacitance of the arrangement by,

$$R = \frac{\varepsilon_0}{\sigma C}.$$



Outline

Electromotive Force (Review)



There are really two forces involved in driving current around a circuit: the source, $\mathbf{f}_s$, and the electrostatic $\mathbf{E}$. Hence the total force per unit charge is given by,

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

Definition

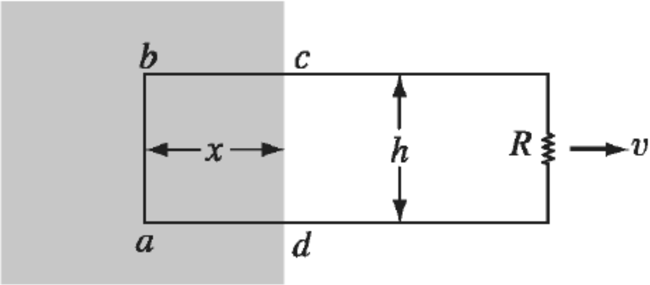
The *electromotive force*, or *emf*, $\mathscr{E}$, of the circuit:

$$\mathscr{E} \equiv \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

Outline

Motional *emf*

A magnetic field **B** pointing into the page.

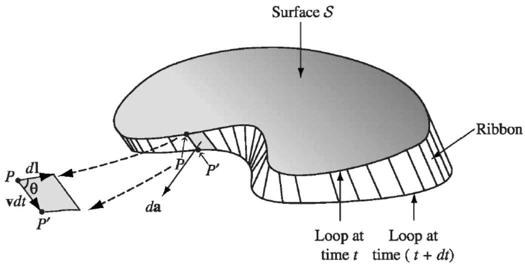


$$\mathscr{E} = \oint \mathbf{f}_{mag} \cdot d\mathbf{l} = Bvh, \qquad \textit{clockwise direction}$$

$\mathscr{E}$ can be evaluated using

$$\mathscr{E} = -\frac{d\Phi}{dt}$$

Motional *emf*



$$d\Phi = \Phi(t+dt) - \Phi(t) = \Phi_{\text{ribbon}} = \int_{\text{ribbon}} \mathbf{B} \cdot d\mathbf{a}$$

$d\mathbf{a} = \mathbf{v}dt \times d\mathbf{l}$, where $\mathbf{v}$ is the wire velocity

If the charge along the wire has velocity $\mathbf{u}$ along the wire, and hence the total charge velocity is $\mathbf{w} = \mathbf{v} + \mathbf{u}$.

$$d\mathbf{a} = \mathbf{v} \times d\mathbf{l}dt = \mathbf{w} \times d\mathbf{l}dt$$

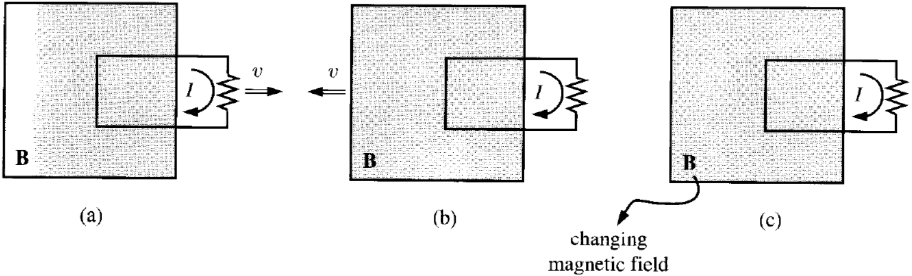
$$\mathcal{E} = -\frac{d\Phi}{dt} = -\oint \mathbf{B} \cdot \mathbf{v} \times d\mathbf{l} = \oint \mathbf{v} \times \mathbf{B} \cdot d\mathbf{l}$$

Outline

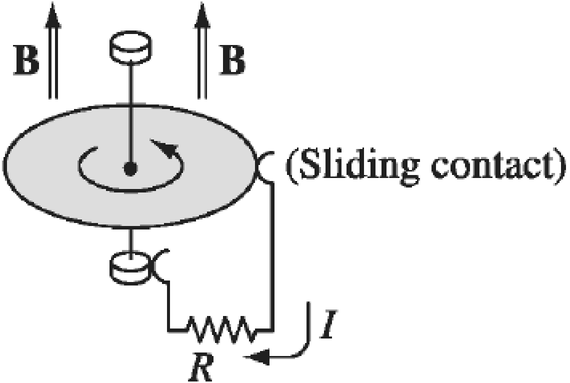
Faraday's Law

A series of experiment made by Micheal Faraday:

- Experiment 1: He pulled a loop of wire to the right through a magnetic field. A current flowed in the loop.
- Experiment 2: He moved the magnet to the left, holding the loop still. Again, a current flowed in the loop.
- Experiment 3: With the loop and the magnet at rest, he changed the strength of the field. Once again, current flowed in the loop.



A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field $\mathbf{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk, as shown in the figure. Find the current in the resistor.



Faraday's Law

$$\mathscr{E} = -\frac{d\Phi}{dt}$$

A changing magnetic field induces an electric field

$$\mathscr{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}'$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Faraday's Law of Electromagnetic Induction

- Fundamental Postulate for Electromagnetic Induction

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

- In integral form,

$$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

Outline

A Stationary Circuit in a Time-Varying Magnetic Field

$$\underbrace{\oint_C \mathbf{E}\cdot d\boldsymbol{\ell}}_{\text{emf induced in circuit with contour } C} = -\frac{d}{dt}\underbrace{\int_S \mathbf{B}\cdot d\mathbf{s}}_{\text{Magnetic flux linked to surface bounded with } C}$$

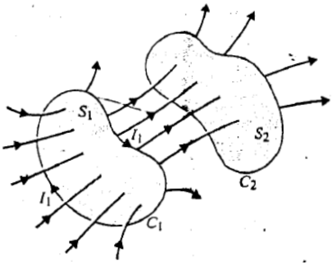
$$\mathscr{V} = -\frac{d\Phi}{dt}$$

Example

A circular loop of N turns of conducting wire lies in the x-y plane with its center at the origin of a magnetic field specified by $\mathbf{B} = \mathbf{a}_z B_0 \cos(\pi r/2b) \sin \omega t$, where b is the radius of the loop and ω is the angular frequency. Find the emf induced in the loop.

Outline

Inductances and Inductors



Mutual flux Φ_{12} is defined as,

$$\Phi_{12} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{s}_2$$

If circuit C_2 has N_2 turns, the *flux linkage* $\Lambda_{12}$is defines as,

$$\Lambda_{12} = N_2 \Phi_{12}$$

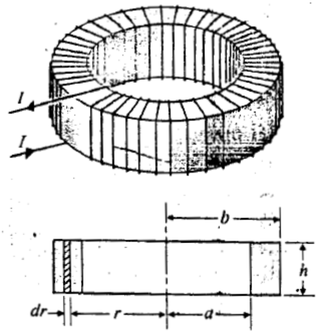
Mutual inductance between circuit C_1 and C_2

$$L_{12} = \frac{\Lambda_{12}}{I_1}$$

Example 1

Example

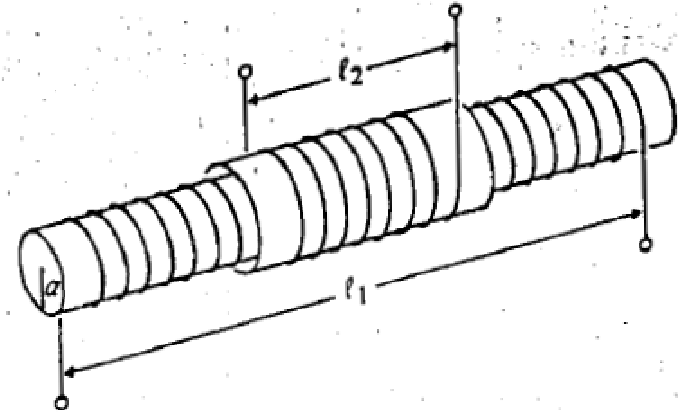
Assume that N turns od wire is tightly wound on a toroidal frame of rectangular cross-section with dimensions as shown in the figure. Then, assuming the permeability of the medium to be μ_0, find the self inductance of the toroidal coil.



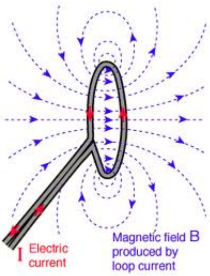
Example 2

Example

Two coils of N_1 and N_2 are wound concentrically on a straight cylindrical core of radius a and permeability μ. The windings have lengths ℓ_1 and ℓ_2, respectively. Finf the mutual inductance between the coils.



Magnetic Energy



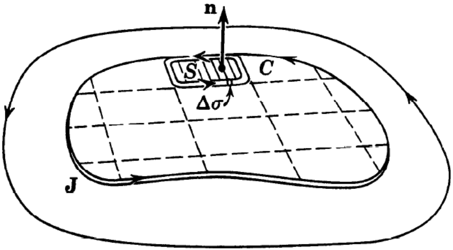
An alternating current will induce emf $v_1 = L_1 \frac{dI}{dt}$. The work required to build the current I_1,

$$W_1 = \int_0^{I_1} v_1 \, i_1 \, dt = \frac{1}{2} L I_1^2$$

So we can prove

$$\boxed{W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k}$$

Magnetic Energy (Field View)



- Field view: Sources do a work W at rate,

$$\frac{dW}{dt} = I\frac{d\Phi}{dt}, \qquad \delta W = I\delta\Phi$$

$$\Delta(\delta W) = J\Delta\sigma \int_S \mathbf{n}\cdot\delta\mathbf{B}ds$$

$$\Delta(\delta W) = J\Delta\sigma \int_S \mathbf{n}\cdot\nabla\times\delta\mathbf{A}ds = J\Delta\sigma \oint_C \delta\mathbf{A}\cdot d\ell$$

$$\delta W = \int_V \mathbf{J}\cdot\delta\mathbf{A}dv$$

$$\delta W = \int_V \nabla\times\mathbf{H}\cdot\delta\mathbf{A}dv, \qquad \longrightarrow \qquad \boxed{\delta W = \int_V \mathbf{H}\cdot\delta\mathbf{B}dv}$$

- For linear media,

$$\boxed{W = \frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}dv}$$

Magnetic Energy

- $W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k$
- $W_m = \frac{1}{2} \int_{V'} \mathbf{A} \cdot \mathbf{J} dv'$
- $W_m = \frac{1}{2} \int_{V'} \mathbf{H} \cdot \mathbf{B} dv'$

Outline

The Induced Electric Field

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

Outline

Maxwell's Equations

1. $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho$, (Gauss law),
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, (Faraday's Law)
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, (Ampere's Law)

We have problem with Ampere's law, for time varying non steady current flow!

A changing electric field induces a magnetic field

Maxwell's added an extra term called the displacement current:

$$\mathbf{J}_d = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}}$$

Maxwell's Equations

1. $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho,$ (Gauss law),
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$ (Faraday's Law)
4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t},$ (Modified Ampere's Law)

Outline

Charges and Currents

- Electrostatic and Magnetostatic bounded charges and currents

$$\rho_b = -\nabla \cdot \mathbf{P}, \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

- polarization current: When there is a time change in the Polarization vector $\mathbf{P}$, there would be a polarization current $\mathbf{J}_p$,

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}.$$

  It satisfies the continuity equation,

$$\nabla \cdot \mathbf{J}_p = -\frac{\partial \rho_b}{\partial t}$$

- Total Charge density ρ,

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

- The total current density $\mathbf{J}$,

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Maxwell's Equations in Matter

- The total charge and current densities ρ and $\mathbf{J}$,

$$\rho = \rho_f - \nabla \cdot \mathbf{P}, \qquad \mathbf{J} = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

- Substiuting in,
  1. $\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho, \qquad$ (Gauss law),
  2. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad$ (Modified Ampere's Law)
- we get the following equations,
  1. $\nabla \cdot \mathbf{D} = \rho_f, \qquad$ (Gauss law),
  2. $\nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}, \qquad$ (Modified Ampere's Law),

  where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

Maxwell's Equations in Matter

1. $\nabla \cdot \mathbf{D} = \rho_f$, (Gauss law),
2. $\nabla \cdot \mathbf{B} = 0$
3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$, (Faraday's Law)
4. $\nabla \times \mathbf{H} = \mathbf{J}_f + \dfrac{\partial \mathbf{D}}{\partial t}$, (Modified Ampere's Law)

where

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \qquad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$$

For linear isotropic materials,

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad \text{and} \quad \mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{D} = \varepsilon \mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

where $\varepsilon = \varepsilon_0 (1 + \chi_e)$, and $\mu = \mu_0 (1 + \chi_m)$.

Displacement Current

$$\nabla \times \mathbf{H} = \underbrace{\mathbf{J}_f}_{\text{Free current}} + \underbrace{\frac{\partial \mathbf{D}}{\partial t}}_{\text{Displacement Current } \mathbf{J}_d}, \qquad \text{(Modified Ampere's Law)}$$

$$\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t}, \qquad \text{Displacement Current}$$

Example

Sea water at frequency $\nu = 4 \times 10^8$ Hz has permittivity $\varepsilon = 81\varepsilon_0$, permeability $\mu = \mu_0$, and conductivity $\sigma = 1/0.23\,\mho/m$. What is the ratio of conduction current to displacement current?
[*Hint*: Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi\nu t)$).]

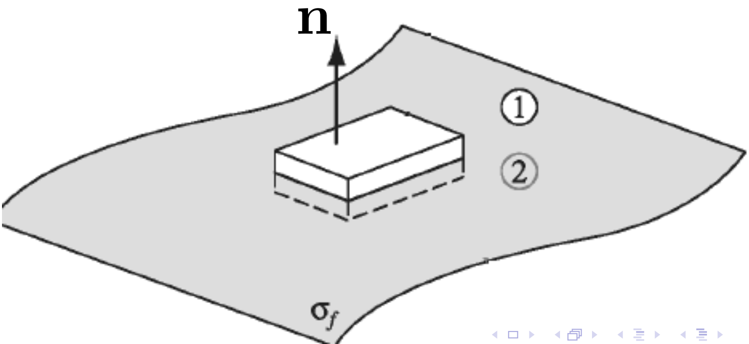
Outline

Boundary Conditions

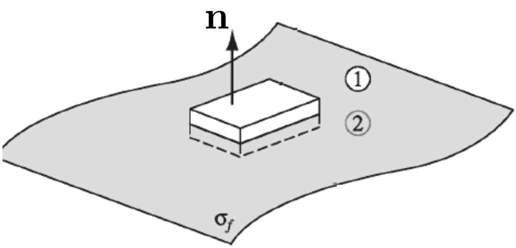
We are going to use the following two theorems,

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{v} \, d\tau = \oint_S \mathbf{n} \cdot \mathbf{v} \, da$$

$$\int_{\mathcal{V}} \nabla \times \mathbf{v} \, d\tau = \oint_S \mathbf{n} \times \mathbf{v} \, da$$



Boundary Conditions



- Tangential Boundary Conditions,

$$E_{1t} = E_{2t}$$

$$\mathbf{n} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{K}_f$$

- Normal Boundary Conditions,

$$\mathbf{n} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \sigma_f$$

$$B_{1n} = B_{2n}$$