Topic 2

High Power Microwave Sources

EEC746

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Equations of Motion

- Non-relativistic

$$m\frac{d\mathbf{u}}{dt} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$

- Relativistic

$$\frac{d\left(\gamma m\mathbf{u}\right)}{dt} = q\left(\mathbf{E} + \mathbf{u} \times \mathbf{B}\right)$$

Outline

Equations of Motion
Energy Equation

The equation of motion for a relativistic particle can be written as,

$$\frac{d}{dt}(\gamma m\mathbf{u}) = q(\mathbf{E}+\mathbf{u}\times\mathbf{B}), \qquad \Longrightarrow \qquad \frac{d}{dt}(\gamma\boldsymbol{\beta}) = \frac{q}{m}\left(\frac{1}{c}\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B}\right)$$

The particle kinetic energy equation can be derived through multiplication with $\boldsymbol{\beta} = \mathbf{u}/c$,

$$\boldsymbol{\beta}\cdot\frac{d}{dt}(\gamma\boldsymbol{\beta}) = \frac{q}{mc}\boldsymbol{\beta}\cdot\mathbf{E}$$

$$\beta^2\frac{d\gamma}{dt} + \frac{\gamma}{2}\frac{d\beta^2}{dt} = \frac{q}{mc}\boldsymbol{\beta}\cdot\mathbf{E}$$

$$\beta^2\frac{d\gamma}{dt} + \frac{1}{\gamma^2}\frac{d\gamma}{dt} = \frac{q}{mc}\boldsymbol{\beta}\cdot\mathbf{E}$$

$$\frac{d\gamma}{dt} = \frac{q}{mc}\boldsymbol{\beta}\cdot\mathbf{E}$$

Energy conservation in static field

Non-relativistic ($u \ll c$)	Relativistic
$m\frac{d\mathbf{u}}{dt} = -e\mathbf{E}$	$\frac{d(\gamma m\mathbf{u})}{dt} = -e\mathbf{E}$, where $\gamma = \frac{1}{\sqrt{1-u^2/c^2}}$

The field $\mathbf{E}$, is given in terms of the potential V, $\mathbf{E} = -\nabla V$.

The energy equation $m\frac{d\mathbf{u}}{dt} = -\frac{e}{mc}\boldsymbol{\beta}\cdot\mathbf{E}$

$$\frac{d(\gamma mc^2)}{dt} = e\mathbf{u}\cdot\nabla V = e\frac{d}{dt}V \quad \Longrightarrow \quad \boxed{\frac{d}{dt}\left(\gamma mc^2 - eV\right) = 0}$$

For non-relativistic equation of motion,

$$\frac{d}{dt}\left(\frac{1}{2}mu^2 - eV\right) = 0, \qquad \Longrightarrow \frac{m}{2}\left(u^2 - u_0^2\right) = e\left(V - V_0\right)$$

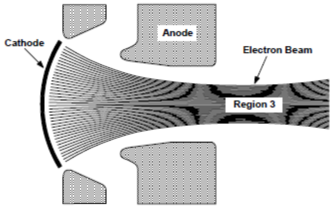
- For an electron with zero initial velocity and zero initial potential, the velocity at a position with potential V is given by,

$$\frac{1}{2}mu^2 - eV = 0, \quad \Longrightarrow \quad u = \sqrt{\frac{2eV}{m}} = \sqrt{2\eta V}$$

Universal Beam Spread Curve

The electron motion in Region 3 can be described by,

$$\frac{d^2 r}{dt^2} = -\eta E_r, \quad \text{where } \eta = \frac{e}{m}$$



The electric field, E_r , produced by the electron charge in a beam is given by,

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = -\frac{Q}{\varepsilon_0} = -\frac{\pi r^2 \ell \rho}{\varepsilon_0}, \qquad \Longrightarrow \qquad E_r = -\frac{r\rho}{2\varepsilon_0}$$

The charge density

$$\rho = \frac{I}{\pi b^2 u_0}, \qquad E_r(r = b) = -\frac{I}{2\pi b u_0}.$$

Hence, the beam envelope equation,

$$\frac{d^2 b}{dt^2} = \frac{\eta I}{2\pi \varepsilon_0 u_0 b}$$

Universal Beam Spread Curve

$$\frac{d^2 b}{dt^2} = \frac{\eta I}{2\pi\varepsilon_0 u_0 b}$$

Writing the equation i terms of z,

$$\frac{db}{dt} = \frac{db}{dz}\frac{dz}{dt}, \qquad \frac{d^2 b}{dt^2} = \frac{d}{dt}\left(\frac{db}{dz}\right)\frac{dz}{dt} + \frac{db}{dz}\frac{d^2 z}{dt^2} = \frac{d^2 b}{dz^2}\left(\frac{dz}{dt}\right)^2 + \frac{db}{dz}\frac{d^2 z}{dt^2}.$$

As $dz/dt = u_0$ and $d^2 z/dt^2 = 0$,

$$\frac{d^2 b}{dz^2} = \frac{\eta I}{2\pi\varepsilon_0 u_0^3 b}$$

Define

$$A = \sqrt{\frac{\eta I}{\pi\varepsilon_0 u_0^3}} = \sqrt{\frac{\eta I}{\pi\varepsilon_0 (2\eta V)^{3/2}}} = \frac{\sqrt{P}}{\sqrt{2\pi\varepsilon_0\sqrt{2\eta}}} = 174\sqrt{P}, \quad \text{where } P = \frac{I}{V^{3/2}}$$

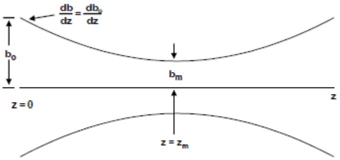
Universal Beam Spread Curve

$$\frac{d^2 b}{dz^2} = \frac{\eta I}{2\pi\varepsilon_0 u_0^3 b}$$

Define

$$A = \sqrt{\frac{\eta I}{\pi\varepsilon_0 u_0^3}} = \sqrt{\frac{\eta I}{\pi\varepsilon_0 (2\eta V)^{3/2}}} = \frac{\sqrt{P}}{\sqrt{\pi\varepsilon_0\sqrt{2\eta}}} = 174\sqrt{P}, \quad \text{where } P = \frac{I}{V^{3/2}}$$

$$\frac{d^2 b}{dz^2} - \frac{A^2}{2b} = 0, \quad \Longrightarrow \quad \frac{d^2\left(\dfrac{b}{b_0}\right)}{d\left(A\dfrac{z}{b_0}\right)^2} - \frac{b_0}{2b} = 0$$

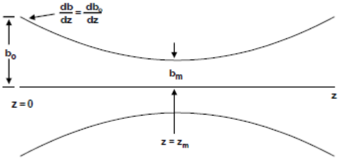


$$\frac{d^2 B}{dZ^2} - \frac{1}{2B} = 0, \qquad \text{where } Z = A\frac{z}{b_0}, \text{ and } B = \frac{b}{b_0}$$

Universal Beam Spread Curve

$$\frac{d^2B}{dZ^2} - \frac{1}{2B} = 0,$$

$$\left(\frac{dB}{dZ}\right)^2 = \ln B + C$$



As $B = 1$ at $Z = 0$, the constant C is given as $(dB_0/dZ)^2$

$$\left(\frac{dB}{dZ}\right)^2 = \ln B + \left(\frac{dB_0}{dZ}\right)^2$$

$$B = e^{(dB/dZ)^2 - (dB_0/dZ)^2}$$

The beam radius reaches minimum, b_m, when $dB/dZ = 0$, which gives, $B_m = e^{-(dB_0/dZ)^2}$.

Universal Beam Spread Curve

$$\left(\frac{dB}{dZ}\right)^2 = \ln B + \left(\frac{dB_0}{dZ}\right)^2$$

$$dZ = \frac{dB}{\left(\ln B + \left(\frac{dB_0}{dZ}\right)^2\right)^{1/2}}, \quad \Longrightarrow \quad Z = \int_1^B \frac{dB}{\left(\ln B + \left(\frac{dB_0}{dZ}\right)^2\right)^{1/2}}$$

Changing the variable of integration from B to

$$u = \frac{dB}{dZ} = \left(\ln B + \left(\frac{dB_0}{dZ}\right)^2\right)^{1/2},$$

$$B = e^{u^2 - (dB_0/dZ)^2}, \qquad dB = 2ue^{u^2 - (dB_0/dZ)^2}\, du$$

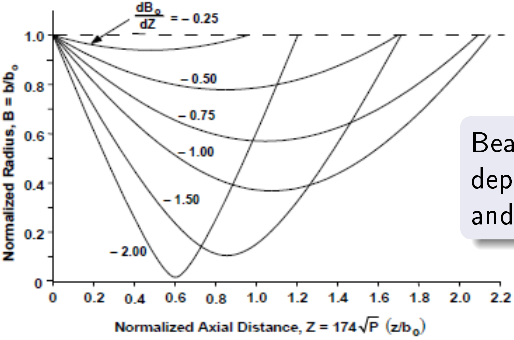
$$Z = 2e^{-(dB_0/dZ)^2} \int_{dB_0/dZ}^{dB/dZ} e^{u^2}\, du$$

Universal Beam Spread Curve

Plots of B as a function of Z can be generated for various values of dB_0/dZ by selecting values of dB/dZ and then calculating corresponding values of B and Z.

$$B = e^{(dB/dZ)^2 - (dB_0/dZ)^2}$$

$$Z = 2e^{-(dB_0/dZ)^2} \int_{dB_0/dZ}^{dB/dZ} e^{u^2}\, du$$



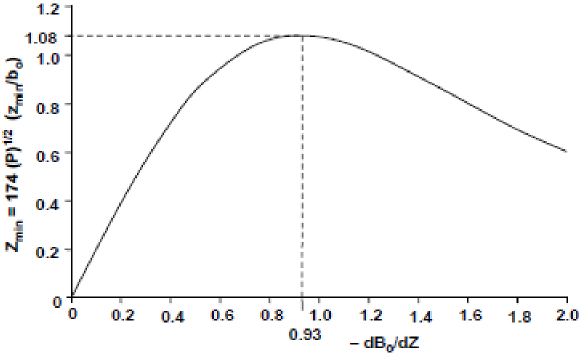
Beam envelope equation only depends on the *initial radius*, *slope* and *beam perveance* P.

Universal Beam Spread Curve

$$Z = 2e^{-(dB_0/dZ)^2} \int_{dB_0/dZ}^{dB/dZ} e^{u^2}\, du$$

Distance for minimum beam radius Z_{min},

$$Z_{min} = 2e^{-(dB_0/dZ)^2} \int_{dB_0/dZ}^{0} e^{u^2}\, du$$



Motion in Cylindrical Coordinates

Transforming from Cartesian to Cylindrical coordinates

$$F_r = F_x \cos\theta + F_y \sin\theta$$

$$F_\theta = -F_x \sin\theta + F_y \cos\theta$$

$$x = r\cos\theta, \qquad y = r\sin\theta$$

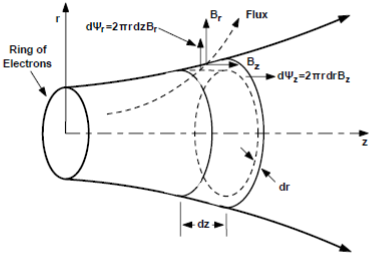
$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta, \qquad \dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

$$\ddot{x} = \left(\ddot{r} - r\dot{\theta}^2\right)\cos\theta - \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\sin\theta, \; \ddot{y} = \left(\ddot{r} - r\dot{\theta}^2\right)\sin\theta + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\cos\theta$$

$$\boxed{F_r = m\left(\ddot{r} - r\dot{\theta}^2\right), \qquad F_\theta = m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)}$$

Outline

Consider the surface described by rotating an electron trajectory about the z-axis.



The magnetic flux Ψ inside this path at a given value of z,

$$\Psi(r,z) = \int_0^r B_z(r',z)\, 2\pi r' dr'$$

$$d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial z} dz = B_z 2\pi r dr + \left[\int_0^r \frac{\partial B_z(r',z)}{\partial z} 2\pi r' dr'\right] dz$$

Busch's Theorem

Using the property $\nabla \cdot \mathbf{B} = 0$, $\quad \Longrightarrow \quad \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0.$

$$\int_0^r \frac{\partial B_z(r',z)}{\partial z} 2\pi r' dr' = -\int_0^r \frac{\partial}{\partial r'}(r' B_r)\, 2\pi dr' = -B_r 2\pi r$$

As,

$$d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial z} dz = B_z 2\pi r dr + \left[\int_0^r \frac{\partial B_z(r',z)}{\partial z} 2\pi r' dr'\right] dz$$

$$\boxed{d\Psi = B_z 2\pi r dr - B_r 2\pi r dz}$$

The force directed in the θ-direction

$$m\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = e\left(\dot{r}B_z - \dot{z}B_r\right)$$

$$m\frac{d}{dt}\left(r^2\dot{\theta}\right) = e\left(r\dot{r}B_z - r\dot{z}B_r\right)$$

$$d\left(r^2\dot{\theta}\right) = \eta\left(rdrB_z - rdzB_r\right)$$

Busch's Theorem

$$d\left(r^2\dot{\theta}\right) = \eta\left(rdrB_z - rdzB_r\right), \qquad \text{and } d\Psi = B_z 2\pi rdr - B_r 2\pi rdz$$

$$\boxed{r^2\dot{\theta} - \frac{\eta}{2\pi}\Psi = \text{Constant of motion}}$$

$$\dot{\theta} = \frac{\eta}{2\pi r^2}\left(\Psi - \Psi_0\right)$$

where Ψ_0 is the flux linked to the path at $\dot{\theta} = 0$ (at the cathode). In most applications of interest in linear-beam tubes, changes slowly with r, so $\Psi = \pi r^2 B_z$.

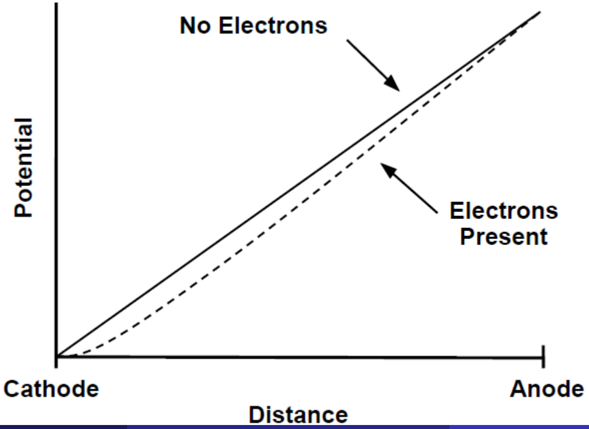
$$\dot{\theta} \approxeq \frac{\eta}{2}\left(B_z - \frac{r_0^2}{r^2}B_{z0}\right)$$

Thermionic Cathodes

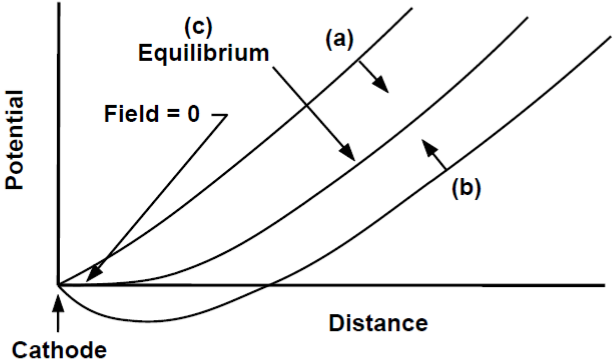
- Two cathode emission mechanisms are used in conventional tubes:
  - Thermionic emission;
  - Secondary emission.
- In cathodes for linear-beam tubes, only thermionic emission is used.
- Secondary emission cathodes are used in crossed-field tubes.
- J. R. Pierce listed the primary characteristics that an ideal cathode should have:
  1. Emits electrons freely, without any form of persuasion such as heating or bombardment (electrons would leak off from it into vacuum as easily as they pass from one metal to another);
  2. Emits copiously, supplying an unlimited current density;
  3. Lasts forever, its electron emission continuing unimpaired as long as it is needed;
  4. Emits electrons uniformly, traveling at practically zero velocity.

Space Charge Limitation

The effect of the negative charge of an electron is to reduce the potential that is present in the absence of the electron. Near an emitting cathode where many electrons are present, the reduction in potential can be appreciable.



Space Charge Limitation

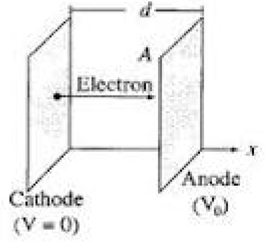


- (a) the electric field at the cathode surface causes all emitted electrons to leave the cathode, thereby depressing the potential near the surface.
- (b) the electric field at the cathode surface forces electrons back to the cathode surface and increases the potential.
- (c) the potential adjacent to the cathode surface is zero, that is, when the electric field at the cathode surface is zero.

Child-Langmuir law

Poisson's equation,

$$\nabla^2 V = -\frac{(-\rho)}{\varepsilon_0} = \frac{\rho}{\varepsilon_0} \qquad \Longrightarrow \qquad \frac{d^2 V}{dx^2} = \frac{\rho}{\varepsilon_0}$$



The charge is given from the current density

$$J = \rho u, \qquad \rho = \frac{J}{u}$$

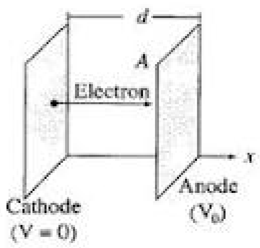
From the conservation of energy,

$$u = \sqrt{2\eta V}, \qquad \Longrightarrow \qquad \rho = \frac{J}{\sqrt{2\eta V}}$$

$$\frac{d^2 V}{dx^2} = \frac{J V^{-1/2}}{\varepsilon_0 \sqrt{2\eta}}$$

$$\frac{1}{2}\left(\frac{dV}{dx}\right)^2 = \frac{2 J V^{1/2}}{\varepsilon_0 \sqrt{2\eta}} \qquad \Longrightarrow \qquad \frac{dV}{dx} = 2\sqrt{\frac{J}{\varepsilon_0 \sqrt{2\eta}}}\, V^{1/4}$$

Child-Langmuir law



$$\frac{dV}{V^{1/4}} = 2\sqrt{\frac{J}{\varepsilon_0\sqrt{2\eta}}}dx, \implies V^{3/4} = \frac{3}{2}\sqrt{\frac{J}{\varepsilon_0\sqrt{2\eta}}}x \quad \implies \quad J = \frac{4}{9}\varepsilon_0\sqrt{2\eta}\frac{V^{3/2}}{x^2}$$

At $x = d$ we have $V = V_0$,

$$I = 2.33 \times 10^{-6}\frac{A}{d^2}V_0^{3/2} = PV^{3/2}$$

$$P = 2.33 \times 10^{-6}\frac{A}{d^2} \qquad \left(\mathrm{A/V}^{3/2}\right)$$

The beam current voltage relation is quite general for any gun geometry.

Electron Gun I-V characteristic

$$I = PV^{3/2}$$

where P is geometrical constant called gun perveance.

Laplace Equation and Analytic Functions

An analytic function has a derivative independent on direction in the complex plane. If $f(z) = f_r(x,y) + if_i(x,y)$ is an analytic function, then

$$\frac{\partial f}{\partial x} = \frac{\partial f_r}{\partial x} + i\frac{\partial f_i}{\partial x}, \qquad \frac{\partial f}{\partial (iy)} = -i\frac{\partial f_r}{\partial y} + \frac{\partial f_i}{\partial y}$$

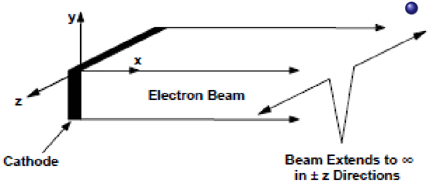
$$\frac{\partial f_r}{\partial x} = \frac{\partial f_i}{\partial y}, \qquad \frac{\partial f_r}{\partial y} = -\frac{\partial f_i}{\partial x}$$

From these equations we can show that

$$\nabla_t^2 f_r = \nabla_t^2 f_i = 0 \qquad \text{(Laplace equation)}$$

The real and imaginary parts of any *complex analytic function* satisfy *Laplace equation* in the two-dimensional plane $x - y$.

Pierce Gun



- In region $y > 0$, no electrons exist and the potential function $V(x, y)$ has to satisfy Laplace equation

$$\nabla_t^2 V(x, y) = 0$$

- In region $y < 0$, it is assumed a one dimensional electron Child-Langmuir space charge limiting current is assumed with potential

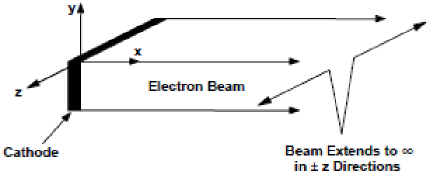
$$V_b(x, y) = \left(\frac{3}{2} \sqrt{\frac{J}{\varepsilon_0 \sqrt{2\eta}}} \right)^{4/3} x^{4/3} = Cx^{4/3}$$

- The continuity of potential has to be satisfied at
  $y = 0, \qquad V(x, 0) = V_b(x, 0)$
- An analytic continuation of the function $Cx^{4/3}$, which is the *only analytic continuation* is,

$$f(x, y) = C\,(x + iy)^{4/3} = f_r(x, y) + if_i(x, y)$$

Both $f_r(x, y)$ and $f_i(x, y)$ satisfy Laplace equation.

Pierce Gun



$$f(x,y) = C\,(x+iy)^{4/3} = f_r(x,y) + i f_i(x,y)$$

- At $y=0$, $f = f_r = Cx^{4/3}$. So we choose the solution $V(x,y)$ as

$$V(x,y) = f_r(x,y) = \Re\{f(x,y)\}$$

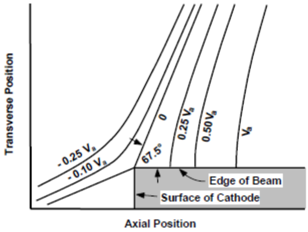
- In Cylindrical coordinates,

$$V(r,\theta) = Cr^{4/3}\cos\left(\frac{4}{3}\theta\right)$$

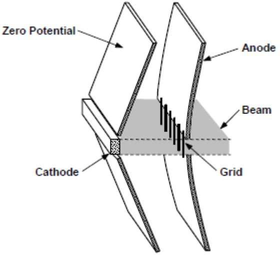
- The zero potential line in $x-y$ plane, is at angle,

$$\theta = \frac{3\pi}{8}\,\text{rad} = 67.5^\circ$$

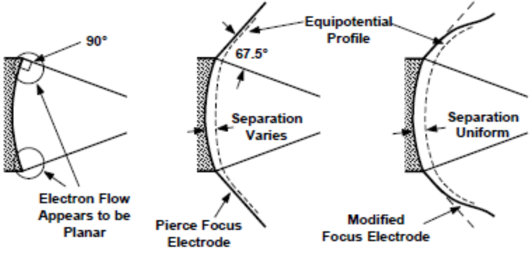
Pierce Gun



Transverse Position
Axial Position
- 0.25 V_a
- 0.10 V_a
67.5°
0
0.25 V_a
0.50 V_a
V_a
Edge of Beam
Surface of Cathode



Zero Potential
Anode
Beam
Cathode
Grid



90°
Electron Flow Appears to be Planar
Equipotential Profile
67.5°
Separation Varies
Pierce Focus Electrode
Separation Uniform
Modified Focus Electrode