

Topic 2

Radiation Field Integrals

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Radiation Integrals

- 1 Maxwell's Equations
- 2 Vector and Scalar Potentials
- 3 Far Field Apprximation

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Instantaneous Fields

$$\nabla \times \mathcal{E}(\mathbf{r}, t) = -\frac{\partial \mathcal{B}(\mathbf{r}, t)}{\partial t}$$

$$\nabla \times \mathcal{H}(\mathbf{r}, t) = \frac{\partial \mathcal{D}(\mathbf{r}, t)}{\partial t} + \mathcal{J}(\mathbf{r}, t)$$

$$\nabla \cdot \mathcal{D} = \rho_t(\mathbf{r}, t)$$

$$\nabla \cdot \mathcal{B} = 0$$

Constitutive relations for free space,

$$\mathcal{D} = \epsilon_0 \mathcal{E}$$

$$\mathcal{B} = \mu_0 \mathcal{H}$$

Phasor Fields

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B}$$

$$\nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

- Constitutive relations

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{B} = \mu \mathbf{H}$$

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Vector and Scalar Potentials

$$\begin{aligned}\nabla \cdot \mathbf{B} = 0, & \quad \Rightarrow \quad \boxed{\mathbf{B} = \nabla \times \mathbf{A}} \\ \nabla \times \mathbf{E} = -j\omega\mathbf{B} & \quad \Rightarrow \quad \nabla \times (\mathbf{E} + j\omega\mathbf{A}) = \mathbf{0} \\ \mathbf{E} + j\omega\mathbf{A} = -\nabla\Phi_e & \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla\Phi_e - j\omega\mathbf{A}}\end{aligned}$$

- Substituting in Maxwell's equation

$$\begin{aligned}\nabla \times \mathbf{H} &= j\omega\epsilon\mathbf{E} + \mathbf{J} \\ \nabla \times \nabla \times \mathbf{A} &= -j\omega\mu\epsilon(\nabla\Phi_e + j\omega\mathbf{A}) + \mu\mathbf{J} \\ \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} &= -j\omega\mu\epsilon\nabla\Phi_e + k^2\mathbf{A} + \mu\mathbf{J} \\ \nabla^2\mathbf{A} + k^2\mathbf{A} &= \nabla(j\omega\mu\epsilon\Phi_e + \nabla \cdot \mathbf{A}) - \mu\mathbf{J}\end{aligned}$$

- One gauge choice is called Lorentz gauge where,

$$j\omega\mu\epsilon\Phi_e + \nabla \cdot \mathbf{A} = 0$$

$$\boxed{\mathbf{E} = -j\omega \left[\mathbf{A} + \frac{1}{k^2} \nabla(\nabla \cdot \mathbf{A}) \right]}$$

Integral Solution of the Fields

- Lorentz gauge

$$j\omega\mu\varepsilon\Phi_e + \nabla \cdot \mathbf{A} = 0$$

- The differential equation for \mathbf{A} ,

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

- Similarly Φ_e satisfies,

$$\nabla^2 \Phi_e + k^2 \Phi_e = -\rho/\varepsilon$$

- Integral Solution for \mathbf{A} , and Φ_e ,

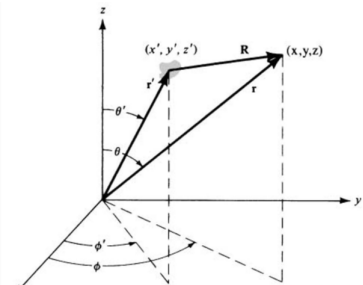


Fig. 3.2(b)

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu}{4\pi} \iiint_V \mathbf{J} \frac{e^{-jkR}}{R} dv'$$

$$\Phi_e(r, \theta, \phi) = \frac{1}{4\pi\varepsilon_0} \iiint_V \rho \frac{e^{-jkR}}{R} dv'$$

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Far Field Apprximation

$$\mathbf{A}(r, \theta, \phi) = \frac{\mu}{4\pi} \iiint_V \mathbf{J} \frac{e^{-jkR}}{R} dv'$$

Far field approximations $r \gg r'$:

$$R = \sqrt{(\mathbf{r} - \mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')} = \sqrt{r^2 + r'^2 - 2\mathbf{r}' \cdot \mathbf{r}}$$

$$R \simeq r - \mathbf{r}' \cdot \hat{\mathbf{a}}_r, \quad kR \simeq kr - \mathbf{k} \cdot \mathbf{r}', \quad \frac{1}{R} \simeq \frac{1}{r},$$

where $\mathbf{k} = k\hat{\mathbf{a}}_r$.

$$\mathbf{A}(r, \theta, \phi) \simeq \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \iiint_V \mathbf{J} e^{j\mathbf{k} \cdot \mathbf{r}'} dv'$$

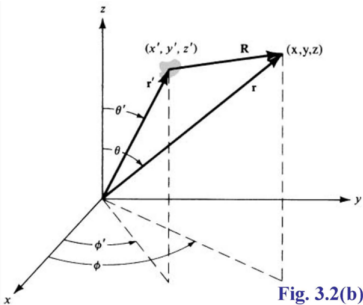


Fig. 3.2(b)

Far Field Apprximation for Magnetic vector Potential

$$\mathbf{A}(r, \theta, \phi) \simeq \frac{e^{-jkr}}{r} \mathbf{A}^0(\theta, \phi), \quad \text{where } \mathbf{A}^0(\theta, \phi) = \frac{\mu}{4\pi} \iiint_V \mathbf{J} e^{j\mathbf{k} \cdot \mathbf{r}'} dv'$$

Integral Solution of the Fields

Far Field approximation

Radiation Magnetic Vector Potential \mathbf{A}

$$\mathbf{A}(r, \theta, \phi) = \frac{e^{-jkr}}{r} \mathbf{A}^0(\theta, \phi), \quad \text{where } \mathbf{A}^0(\theta, \phi) = \frac{\mu_0}{4\pi} \iiint_V \mathbf{J} e^{j\mathbf{k} \cdot \mathbf{r}'} dv'$$

$$\nabla \times \mathbf{A} = \nabla \left(\frac{e^{-jkr}}{r} \right) \times \mathbf{A}^0(\theta, \phi) + \frac{e^{-jkr}}{r} \nabla \times \mathbf{A}^0(\theta, \phi)$$

$$\begin{aligned} \nabla \times \mathbf{A}^0 &= \frac{\mathbf{a}_r}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\phi^0 \sin \theta) - \frac{\partial A_\theta^0}{\partial \phi} \right] + \frac{\mathbf{a}_\theta}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r^0}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi^0) \right] \\ &\quad + \frac{\mathbf{a}_\phi}{r} \left[\frac{\partial}{\partial r} (r A_\theta^0) - \frac{\partial A_r^0}{\partial \theta} \right] = \mathcal{O} \left(\frac{1}{r} \right) \end{aligned}$$

$$\nabla \times \mathbf{A} = \nabla \left(\frac{e^{-jkr}}{r} \right) \times \mathbf{A}^0(\theta, \phi) + \frac{e^{-jkr}}{r} \nabla \times \mathbf{A}^0(\theta, \phi) \quad \mathcal{O} \left(\frac{1}{r^2} \right)$$

Far Field Approximation of the Fields

$$\nabla \times \mathbf{A} = \nabla \left(\frac{e^{-jkr}}{r} \right) \times \mathbf{A}^0(\theta, \phi) + \frac{e^{-jkr}}{r} \nabla \times \mathbf{A}^0(\theta, \phi) \quad \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$\nabla \times \mathbf{A} = -jk \frac{e^{-jkr}}{r} \hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta, \phi) - \frac{e^{-jkr}}{r^2} \hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta, \phi) \quad \mathcal{O}\left(\frac{1}{r^2}\right)$$

$$+ \frac{e^{-jkr}}{r} \nabla \times \mathbf{A}^0(\theta, \phi) \quad \mathcal{O}\left(\frac{1}{r^2}\right)$$

Far Field Approximation of $\nabla \times \mathbf{A}$

$$\nabla \times \mathbf{A} = -jk \hat{\mathbf{a}}_r \times \mathbf{A}, \quad \text{where } \mathbf{A}(r, \theta, \phi) = \frac{e^{-jkr}}{r} \mathbf{A}^0(\theta, \phi)$$

Far Field Approximation of the Fields

Far Field Approximation of $\nabla \times \mathbf{A}$

$$\nabla \times \mathbf{A} = -jk\mathbf{a}_r \times \mathbf{A}, \quad \text{where } \mathbf{A}(r, \theta, \phi) = \frac{e^{-jkr}}{r} \mathbf{A}^0(\theta, \phi)$$

- The magnetic field \mathbf{H} is approximated in the far field as,

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \simeq \frac{-jk}{\mu} \mathbf{a}_r \times \mathbf{A} = -j\omega \sqrt{\frac{\epsilon}{\mu}} \frac{e^{-jkr}}{r} \mathbf{a}_r \times \mathbf{A}^0(\theta, \phi)$$

$$\mathbf{H} = \frac{e^{-jkr}}{r} \mathbf{H}^0(\theta, \phi), \quad \text{where } \mathbf{H}^0(\theta, \phi) = -j\omega \sqrt{\frac{\epsilon}{\mu}} \mathbf{a}_r \times \mathbf{A}^0(\theta, \phi)$$

- The electric field \mathbf{E} far away from the source where $\mathbf{J} = 0$, satisfies,

$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$$

- In the far field region the electric field can be approximated as,

$$\mathbf{E} = \frac{1}{j\omega\epsilon} \nabla \times \mathbf{H} \simeq \frac{-jk}{j\omega\epsilon} \mathbf{a}_r \times \mathbf{H} = -\sqrt{\frac{\mu}{\epsilon}} \mathbf{a}_r \times \mathbf{H} = j\omega$$

Far Field Approximation of the Fields

- The magnetic field \mathbf{H} is approximated in the far field as,

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \simeq \frac{-jk}{\mu} \hat{\mathbf{a}}_r \times \mathbf{A} = \frac{-j\omega}{\eta} \hat{\mathbf{a}}_r \times \mathbf{A} = \frac{-j\omega}{\eta} \frac{e^{-jkr}}{r} \hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta, \phi)$$

$$\mathbf{H} = \frac{e^{-jkr}}{r} \mathbf{H}^0(\theta, \phi), \quad \text{where } \mathbf{H}^0(\theta, \phi) = -j\omega \sqrt{\frac{\epsilon}{\mu}} \hat{\mathbf{a}}_r \times \mathbf{A}^0(\theta, \phi)$$

- In the far field region the electric field \mathbf{E} can be approximated as,

$$\mathbf{E} = -\eta \hat{\mathbf{a}}_r \times \mathbf{H} = j\omega \hat{\mathbf{a}}_r \times (\hat{\mathbf{a}}_r \times \mathbf{A})$$

- Using the vector identity, $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$,

$$\mathbf{E} = j\omega \hat{\mathbf{a}}_r \times (\hat{\mathbf{a}}_r \times \mathbf{A}) = j\omega [A_r \hat{\mathbf{a}}_r - \mathbf{A}] = -j\omega (A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi)$$

Far Field Approximation of the Fields

Conclusion

Electric Field \mathbf{E}

$$\mathbf{E} = -j\omega (A_\theta \hat{\mathbf{a}}_\theta + A_\phi \hat{\mathbf{a}}_\phi)$$

$$\mathbf{E} = \frac{e^{-jkr}}{r} \mathbf{E}^0(\theta, \phi), \text{ where } \mathbf{E}^0 = E_\theta^0 \hat{\mathbf{a}}_\theta + E_\phi^0 \hat{\mathbf{a}}_\phi$$

Magnetic Field \mathbf{H}

$$\mathbf{H} = \frac{1}{\eta} \hat{\mathbf{a}}_r \times \mathbf{E}$$

$$\mathbf{H} = \frac{e^{-jkr}}{r} \mathbf{H}^0(\theta, \phi), \text{ where } \mathbf{H}^0(\theta, \phi) = \frac{1}{\eta} \hat{\mathbf{a}}_r \times \mathbf{E}(\theta, \phi)$$

In the far field the radiation fields constitute spherical TEM wave.

Integral Solution of the Fields

Far Field approximation

$$\mathbf{H} = \frac{1}{\mu} \nabla \times \mathbf{A} \simeq -j\omega \sqrt{\frac{\epsilon}{\mu}} \frac{e^{-jkr}}{r} \mathbf{a}_r \times \mathbf{A}^0(\theta, \phi)$$

Radiation Electric and Magnetic Fields

$$\mathbf{H} \simeq \frac{1}{\eta} \mathbf{a}_r \times \mathbf{E}, \quad \mathbf{E} \simeq -j\omega [A_\theta \mathbf{a}_\theta + A_\phi \mathbf{a}_\phi]$$

$$\mathbf{A}(r, \theta, \phi) \simeq \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \iiint_V \mathbf{J}(r', \theta', \phi') e^{j\mathbf{k} \cdot \mathbf{r}'} dv'$$

- Surface current

$$\mathbf{A}(r, \theta, \phi) \simeq \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \iint_S \mathbf{J}_s(r', \theta', \phi') e^{j\mathbf{k} \cdot \mathbf{r}'} ds'$$

- Linear current

$$\mathbf{A}(r, \theta, \phi) \simeq \frac{\mu}{4\pi} \frac{e^{-jkr}}{r} \int_C I_e(r', \theta', \phi') e^{j\mathbf{k} \cdot \mathbf{r}'} dl'$$

