Continuous Time Markov Chains (CTMC)
Lecture #8

Contents

1 Markov Process (Continuous Time Markov Chain) 1
1.1 Transient Analysis of CTMC . . . 1
1.1.1 Pure Birth Transient Analysis . . . 1
1.1.2 Pure death Transient Analysis . . . 2
2 M/G/1 Queue Length Analysis 3
3 Homework 5

1 Markov Process (Continuous Time Markov Chain)

1.1 Transient Analysis of CTMC

Generally, transient analysis is a very challenging task. However, instantaneous state probability can be driven for a few case especially when the state probabilities at time 0, $\pi_i(0)$, are known. E.g. usually one knows that the system at time 0 is precisely in a given state k. The objective of the transient analysis is to find how the state probabilities evolve as a function of time π(t)

The transient dynamics of CTMC are mainly defined by

$$\frac{d}{dt}\pi(t) = \pi(t)Q \tag{1}$$

1.1.1 Pure Birth Transient Analysis

It is assumed that the system starts with zero population, that is to say $\pi_0(0) = 1$ and $\pi_i(0) = 0 \ \forall i \neq 0$.

Using eq(1), one can write

$$\frac{d}{dt}\pi_o(t) = -\lambda\pi_o(t)$$

$$\frac{d}{dt}\pi_i(t) = -\lambda\pi_i(t) + \lambda\pi_{i-1}(t)$$

Solving these differential equations using laplace transform, we have

$$s\pi_o^*(s) - \pi_0(0) = -\lambda\pi_o^*(s) \rightarrow \pi_o^*(s) = \frac{1}{s+\lambda}$$

$$\pi_i^*(s) = \frac{\lambda}{s+\lambda}\pi_{i-1}^*(s) = \frac{\lambda^k}{(s+\lambda)^{k+1}}$$

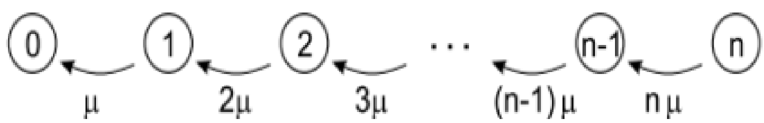
Applying laplace transform, one can express the instantensous probabilites as

$$\pi_o(t) = e^{-\lambda t}$$

$$\pi_i(t) = \frac{(\lambda t)^i}{i!}e^{-\lambda t}$$

Note that the obtained solution indicates that the population in some period t follows a Poisson process.

1.1.2 Pure death Transient Analysis



It is assumed that the system starts with a population of n members, that is to say $\pi_n(0) = 1$ and $\pi_i(0) = 0 \ \forall i \neq n$

Using eq(1), one can write

$$\frac{d}{dt}\pi_n(t) = -n\mu\pi_n(t) \Rightarrow \pi_n(t) = e^{-n\mu t}$$

$$\frac{d}{dt}\pi_i(t) = (i+1)\mu\pi_{i+1}(t) - i\mu\pi_i(t) \Rightarrow \frac{d}{dt}\left(e^{i\mu t}\pi_i(t)\right) = (i+1)\mu\pi_{i+1}(t)e^{i\mu t}$$

Hence, one can express $\pi_i(t)$ as

$$\pi_i(t) = (i+1)\mu e^{-i\mu t}\int_0^t \pi_{i+1}(\tau)e^{i\mu\tau}d\tau$$

Solving this equation for $i = n-1$, we have

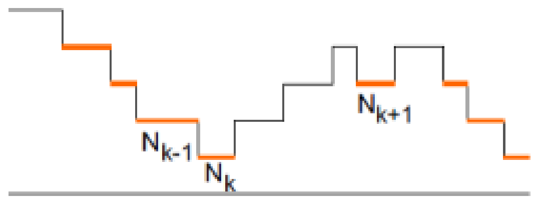
$$\begin{aligned}\pi_{n-1}(t) &= n\mu e^{-(n-1)\mu t}\int_0^t e^{-n\mu\tau} * e^{(n-1)\mu\tau}d\tau \\ &= n\mu e^{-(n-1)\mu t}\int_0^t e^{-\mu\tau}d\tau \\ &= n\left[(e^{-\mu t})^{n-1}\right]\left[1-e^{-\mu t}\right]\end{aligned}$$

Solving recursively for $n-2$, $n-3$, ... one can express the transient state probabilities as

$$\pi_i(t) = \left(\begin{array}{c} n \\ i \end{array}\right)\left[(e^{-\mu t})\right]^i\left[1-e^{-\mu t}\right]^{n-1}$$

2 M/G/1 Queue Length Analysis

- The analysis is based on Embedded Markov chain or jump chain
- Let N_k be the queue length after the departure of customer k
- Let V_k be the number of new customers arrived during the service time of customer k.



- Markov chain is constituted by the queue left by an departing customer.
  - Given N_k , N_{k+1} can be expressed in terms of it and of a random variable V_{k+1}

$$N_{k+1} = \begin{cases} N_k - 1 + V_{k+1} & , N_k \geq 1 \\ V_{k+1} & , N_k = 0 \end{cases}$$

  - As the service times are independent and the arrivals are Poissonian, the V_k are independent of each other.
  - V_{k+1} is independent of N_k and its history
  - The stochastic characterization of N_{k+1} depends on N_k but not on the earlier history $\rightarrow$ Markov Process

- Let a_j denotes the probability of j arrivals between two departures, i.e. $P(V_k = j) = a_j$
- Let p_j denotes the length distribution probability, then

$$p_j = p_0 a_j + \sum_{i=1}^{j+1} p_i a_{j-i+1}$$

- The MGF for the number of the customers in the system can then be derived as

$$\begin{aligned} G_N(z) &= \sum_{j=0}^{\infty} z^j p_j = \sum_{j=0}^{\infty} z^j a_j p_o + \sum_{j=0}^{\infty} \sum_{i=1}^{i=j+1} z^j p_i a_{j-i+1} \\ &= p_o G_V(z) + \sum_{i=1}^{\infty} \sum_{j=i-1}^{\infty} z^j p_i a_{j-i+1} \\ &= p_o G_V(z) + \sum_{i=1}^{\infty} p_i \sum_{k=0}^{\infty} z^{k+i-1} a_k \\ &= p_o G_V(z) + \sum_{i=1}^{\infty} p_i z^{i-1} \sum_{k=0}^{\infty} z^k a_k \\ &= p_o G_V(z) + z^{-1}[G_N(z) - p_o] G_V(z) \end{aligned}$$

Hence, one can conclude that

$$G_N(z) = \frac{(z-1)p_o G_V(z)}{z - G_V(z)} \tag{2}$$

Eq (2) indicates that if one can characterize the number of arrivals between two departures $[G_v(z)]$, one can characterize the limiting distribution of the number of the customers in the system.

- Note that $G_N(1) = 1 = \frac{(1-1)p_o G'_V(1) + p_o G_V(1)}{1 - G'_V(1)} \rightarrow p_o = 1 - G'(1) = 1 - \rho$
- In the following, we charactrize the distribution of inter-departure arrival
  - First, let us characterize the number of arrivals from a Poisson process during a random interval X

$$\begin{aligned} G_V(z) &= \sum_{v=0}^{\infty} z^v P\{V = v\} \\ &= \sum_{v=0}^{\infty} z^v \int_0^{\infty} P\{V = v | S = s\} f_S(s) ds \\ &= \sum_{v=0}^{\infty} z^v \int_0^{\infty} \frac{(\lambda s)^v}{v!} e^{-\lambda s} f_S(s) ds \\ &= \int_0^{\infty} \sum_{v=0}^{\infty} \frac{(z\lambda s)^v}{v!} e^{-\lambda s} f_S(s) ds \\ &= \int_0^{\infty} e^{z\lambda s} e^{-\lambda s} f_S(s) ds \\ &= \int_0^{\infty} e^{-(1-z)\lambda s} f_S(s) ds \\ &= G_S(\lambda(1-z)) \end{aligned} \tag{3}$$

  - The same result can be attained using the law of iterated expectations

$$G_V(z) = E_V[z^k] = E_S[E_{V|S}[z^k|S]] = E_S[e^{-(1-z)\lambda S}] = G_S((1-z)\lambda)$$

- By pluging (3) in (2), we have

$$G_N(z) = \frac{(z-1)p_o G_S(\lambda(1-z))}{z - G_S(\lambda(1-z))}$$

- For M/M/1, we have $G_S(s) = \frac{\mu}{\mu + s} \Rightarrow G_S((1-z)\lambda) = \frac{\mu}{\mu + (1-z)\lambda} = 1/(1 + (1-z)\rho)$

$$\begin{aligned} G_N(z) &= \frac{(z-1)(1-\rho)}{z(1 + (1-z)\rho) - 1} = \frac{1-\rho}{1-\rho z} \\ &= (1-\rho)(1 + \rho z + (\rho z)^2 + ...) \end{aligned}$$

3 Homework

- Consider a system with two components whose failure rate is λ. The component repair time is exponentially distributed with rate μ. However, the system fails if both components fail. Fine the mean time to failure of such system. Characterize the failure time of this system.