Markov Chains

Lecture #6

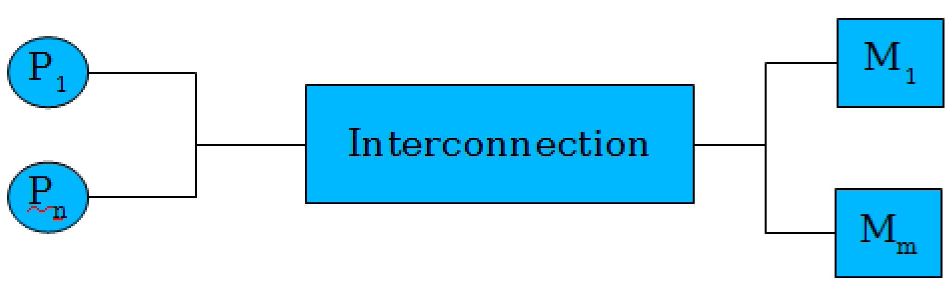
Contents

1 DTMC

1.1 Steady State Analysis

1.1.1 Example: Multi Processor Memory Utilization

- M memory modules shared by N processors
- each processor may request memory module i with probability q_i
- Each memory access consumes a complete time slot
- Each processor is fast enough to request a new slot once it finishes its ongoing access
- If two or more processors request the same memory module, the different requests are enqueued
- We need to study the memory utilization??



Solution

- Let the state defined by the number of processor waiting to be served by module i every slot
- For $m = 2$ and $n = 2$

$$P = \begin{array}{c|ccc} & (1,1) & (0,2) & (2,0) \\ \hline (1,1) & 2q_1q_2 & q_2^2 & q_1^2 \\ (0,2) & q_1 & q_2 & 0 \\ (2,0) & q_2 & 0 & q_1 \end{array}$$

- By solving the problem, one can derive the follwoing relations $\pi_2 = \frac{q_1^2}{1-q_1}\pi_o$ and $\pi_1 = \frac{q_2^2}{1-q_2}\pi_o$
- With some algebraic manipulation, we can estimate $\pi_o = \frac{q_1q_2}{1-2q_1q_2}$
- We can estimate the expected memory utilization, Z, as

$$E[Z] = 2 * \pi_0 + \pi_1 + \pi_2 = (1 - q_1q_2)/(1 - 2q_1q_2)$$

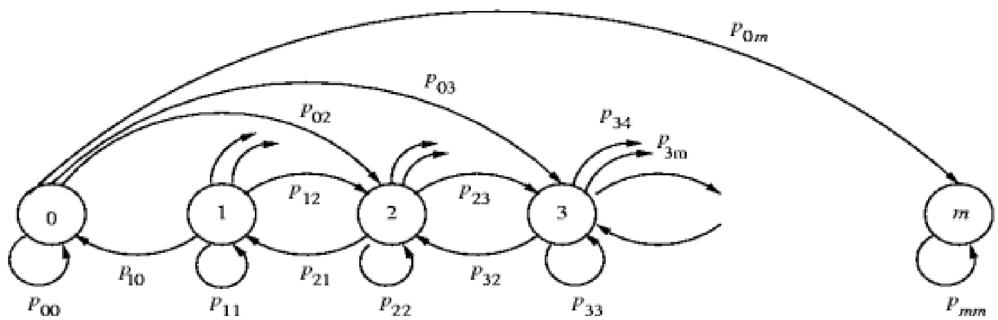
- The maximum memory utilization occurs when $q_1 = q_2 = 0.5$

1.1.2 Example: Slotted Aloha Analysis

- The ALOHAnet uses a new method of medium access (ALOHA random access)
- In slotted Aloha, a station can send only at the beginning of a timeslot, and thus collisions are reduced.
- The system has m users.
- If two (or more) users collide, they are enqueued as backlogged users
- New users access the channel with prob a
- Backlogged users access the medium with prob. b
- Our objective is to analyze the number of backlogged users in the systes as a quality metric.

Solution

- Successful channel access if:
  - Exactly one new req. and no backlogged req.
  - Exactly one backlogged req. and no new req.
- Unsuccessful transmission increases the number of backlogged users
- The number of backlogged users represents a Markov chain



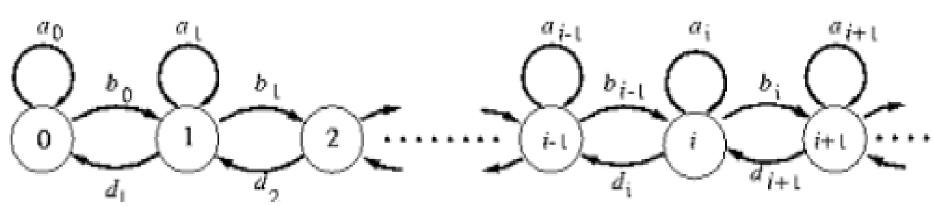
- Let we have n backlogged users and m-n new users
- Let A(i,n) denotes the prob. that i-new users try to access the channel. Let B(i,n) denotes the prob. that i-backlogged users try to access the channel. The state transition probabilities can be expressed as

$$p_{n,n+i} = \begin{cases} A(i,n), & 2 \leq i \leq m-n \\ A(1,n)[1-B(0,n)], & i = 1 \\ A(1,n)B(0,n) + A(0,n)[1-B(1,n)], & i = 0 \\ A(0,n)B(1,n), & i = -1 \end{cases}$$

$$A(i,n) = \binom{m-n}{i} a^i (1-a)^{m-n-i}, \qquad 0 \leq i \leq m-n$$

$$B(i,n) = \binom{n}{i} b^i (1-b)^{n-i} \qquad n \geq i \geq 0$$

1.2 Discrete-time Birth-Death Process



$$p_{i,i+1} = b_i \;\; \textit{birth probablity}$$

$$p_{i,i-1} = d_i \;\; \textit{death probability}$$

$$p_{ii} = a_i \quad \textit{no change}$$

- The steady state solution can be determined using $\pi P = \pi$ together with the normaliztion equation $\sum_i \pi_i = 1$.
- Another approach to dolve this problem is to benefit from the balance equation by defining ***cut sets*** *and check the flow among these state sets.*

$$\begin{aligned} \pi_i &= \frac{b_{i-1}}{d_i} v_{i-1} \\ &= \underbrace{\left[\prod_{k=1}^{i} \frac{b_{k-1}}{d_k}\right] \pi_o}_{by\, recursive\, application} \end{aligned}$$

and substituting in the normalization equation one can get

$$\pi_o = \left(\sum_i \left[\prod_{k=1}^{i} \frac{b_{k-1}}{d_k}\right]\right)^{-1}$$

1.2.1 Example: Data Structure Memory Utilization

- Analyze memory usage for a specific program
  - $b_i \rightarrow$ probability of insertion in the data structure
  - $d_i \rightarrow$ probability of deletion from the data structure
  - $a_i \rightarrow$ probability of access to the data structure

Solution

- $\pi_i \rightarrow$ probability of i-data elements in the memory
- $\pi_i = [1 - (b/d)](b/d)^i]$
- Modified geometric with parameter $[1 - (b/d)]$
- Average memory utilization will be (b/d) / [1-(b/d)]

1.2.2 Example: Data structure with limited memory

- Let the memory space can store m-elements

$$\pi_i = 1/\sum_{i=1}^{m} (b/d)^i = \frac{1 - b/d}{1 - (b/d)^{m+1}}$$

- Hence, Memory overflow probability is

2 Markov Chains with Absorbing state(s)

- Absorbing states usually model the end
  - e.g. end of a program execution
  - e.g. a complete failure of the system.

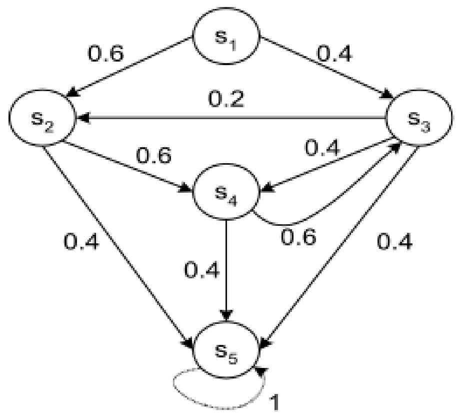


Figure 1: A program having a set of interacting modules with an exit absorbing state

- The Canonical Form representation for a DTMC with r absorbing states and t transient states is expressed as
$$P = \begin{pmatrix} \mathbf{Q}_{(n-r)x(n-r)} & \mathbf{R}_{(n-r)x(r)} \\ \mathbf{0}_{rx(n-r)} & \mathbf{I}_{rxr} \end{pmatrix}$$
where I is the identity matrix, 0 is a zero matrix, R is a representing the transition probabilities to the absorbing states, and $\mathbf{Q}$ is an matrix identifying the transition probabilities among transient states

- The **n-step** transition probability Matrix can be expressed as
$$P^n = \begin{pmatrix} \mathbf{Q^n} & N\mathbf{R} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$
where $N = (I + Q + Q^2 + ... + Q^{n-1}) = (I - Q)^{-1}$ and is called the fundmental matrix of Markov chain with absorbing states.

- Note also that $Q^n \to 0$ as $n \to \infty$ (why?)

- Note also that as $n \to \infty$, NR represents the **absorption probabilities** to different states. What does the item $[NR]_i$ represent?

> **Theorem**
> The entry n_{ij} of N gives the expected number of times that the process is in the transient state s_j if it is started in the transient state s_i .

proof
Let $X^{(k)}$ be a random variable which equals 1 if the chain is in state s_j after k steps, and equals 0 otherwise.
Hence, $P(X^{(k)} = 1) = p_{ij}^{(k)}$ and $P(X^{(k)} = 0) = 1 - p_{ij}^{(k)} \Rightarrow E\left[X^{(k)}\right] = p_{ij}^{(k)}$
Hence, expected number of visits to s_j is $E[X^{(0)} + X^{(1)} + ... + X^{(n)}] = p_{ij}^{(0)} + p_{ij}^{(1)} + ... + p_{ij}^{(n)}$.
Note that $p_{ij}^{(0)} = \pi_j(0)$.
AS $n \to \infty$, $E[X^{(0)} + X^{(1)} + ... + X^{(n)}] = p_{ij}^{(0)} + p_{ij}^{(1)} + ... = n_{ij}$
Note that n_{ij} is the conditional number of visits assuming that the chain started in state i. Hence, the unconditional number of visits for S_j can be estimated using the total probability theorem . $\sum_j \pi_i(0) n_{ij}$.

- Let t_i be the expected number of steps before the chain is absorbed given that the chain starts in state s_i , and let t be the column vector whose i^{th} entry is t_i. Then $t = N\mathbf{e}$, where $\mathbf{e}$ is a column vector all of whose entries are 1.

3 Homework

- An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability p each time she commutes, independently of other times. What is the steady-state probability that she gets wet on a given day?

- A fly moves along a straight line in unit increments. At each time period, it moves one unit to the left with probability 0.3, one unit to the right with probability 0.3, and stays in place with probability 0.4, independently of the past history of movements. A spider is lurking at positions 1 and a frog is lurking at position m: if the fly lands in any of these end positions, it is captured by the spider or the frog, and the process terminates. Determine the probability that the fly is eaten by the spider.