Important Concepts

Lecture #3

Contents

1 Sum of Random Variables

1.1 Sum of Deterministic Number of RVs

- naturally appear in many applications with subsequent operations.
- For example, calculating the total delay over multiple links, hard-disk search delay
- Let $W = X + Y$, where X and Y are independent RVs, the the PDF of Z can be expressed as

$$\begin{aligned}
F_W(w) &= P(W \leq w) \\
&= P(X + Y \leq w) \\
&= \int_{x=-\infty}^{\infty} \int_{y=-\infty}^{w-x} f_X(x) f_Y(y) dy dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) F_Y(w-x) dx
\end{aligned}$$

$$\begin{aligned}
f_W(w) &= \frac{dF_W}{dw}(w) \\
&= \frac{d}{dw} \int_{x=-\infty}^{\infty} f_X(x) F_Y(w-x) dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) \frac{d}{dw} F_Y(w-x) dx \\
&= \int_{x=-\infty}^{\infty} f_X(x) f_Y(w-x) dx
\end{aligned}$$

1.2 Important Theorems

Theorem 1 The sum of "r" mutually identical independent exponentially distributed RV is an "r" stage Erlang distribution with parameter λ

Theorem 2 The sum of "r" mutually independent exponentially distributed RVs follows a hypo-exponential distribution

Theorem 3 If X_1, X_2, .., X_k are normal 'IID' RV's, then, the RV $Z = (X_1 + X_2 + .. + X_k)$ is also normal with

$$\overline{Z} = \sum_{i=1}^{k} \overline{X_i} \ \ and \ \ \sigma_Z^2 = \sum_{i=1}^{k} \sigma_{X_i}^2$$

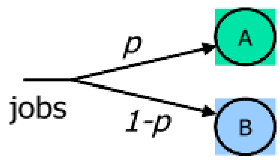
Theorem 4 If X_1, X_2, .., X_k are normal 'IID' RV's, then, the RV $Y = \sum_{i=1}^{k} X_i^2$ follows Gamma distribution $(\frac{1}{2}, \frac{k}{2})$

2 Conditional Distribution

- Given two jointly distributed random variables X and Y, the conditional probability distribution of Y given X is the probability distribution of Y when X is known to be a particular value. If the conditional distribution of Y given X is a continuous distribution, then its probability density function is known as the ***conditional density function***.
- Conditional density for Modeling: When constructing probabilistic models for experiments that have a sequential character, it is often natural and convenient to first specify conditional probabilities and then use them to determine unconditional probabilities.

Example

A two-server system is receiving jobs according to a Poisson arrival process. The jobs are assigned to server A with probability p and to server B with probability (1-p) as shown in Figure. Derive an expression for the distributions assigned to server A.



Solution

Let RV N represents the number of jobs that arrive at the system
Let RV K denotes the number of jobs assigned to server A
Assume that the system receives exactly n jobs. Hence, one can express the distribution of the Server A jobs as

$$P_{K|N=n}(k|N=n) = \binom{n}{k} p^k (1-p)^{n-k}$$

Noting that N~Poisson, $P(N=n) = \frac{\lambda^n}{n!}e^{-\lambda}$
Consequently, on can estimate the distribution of K as

$$\begin{aligned} p_K(k) &= \sum_n P_{K|N=n}(k|N=n)P(N=n) \\ &= \sum_n \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} e^{-\lambda} \\ &= \frac{(p\lambda)^k}{k!} e^{-p\lambda} \end{aligned}$$

Example

System with two components in series. Each has an exponential failure time with parameter λ_i, i=1,2. Find the probability that the second component causes the system failure.

Solution

Let the second component fails at time $T_2 = t_2$
The probability we are considering is that

$$\begin{aligned} P(T_1 &> T_2) = \int_{-\infty}^{\infty} e^{-\lambda_1 t_2} f_{T_2}(\tau) d\tau \\ &= \int_{0}^{\infty} e^{-\lambda_1 t_2} \lambda_2 e^{\lambda_2 \tau} d\tau \\ &= \frac{\lambda_2}{\lambda_2 + \lambda_1} \end{aligned}$$

3 Conditional Expectation as a Random Variable

The value of the conditional expectation $E[X|Y = y]$ of a random variable X given another random variable Y depends on the realized experimental value y of Y. Hence, $E[X|Y]$ a function of Y , and therefore a random variable. For this new RV, we are interested in the estimation of its mean and variance

Recall the fact that

$$E[X|Y = y] = \begin{cases} \sum_x x p_{X|Y}(x|y) & , \textit{discrete} \\ \int_{-\infty}^{\infty} x f_{X|Y}(x|y) & , \textit{continuous} \end{cases}$$

Once a value of y is given, the above summation or integration yields a numerical value for $E[X|Y = y]$.

$$E[E[X|Y]] = \begin{cases} \sum_y E[X|Y = y] p_Y(y) & \textit{discrete} \\ \int_{-\infty}^{\infty} E[X|Y = y] f_Y(y) dy & , \textit{continuous} \end{cases}$$

Since $E[X|Y]$ is a random variable, it has an expectation $E[E[X|Y]]$ that can be expressed as Both expression would dissolve to $E[X]$ by total probability theorem. This result is known as ***Law of Iterated Expectation***

Law of Iterated Expectation

$$E[X] = E[E[X|Y]]$$

Example

We start with a stick of length l. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the stick that we were left with. What is the expected length of the stick that we are left with, after breaking twice?

Solution

$E[X] = E[E[X|Y]] = E[Y/2] = E[Y]2 = l/4$

The conditional distribution of X given $Y = y$ has a mean, which is $E[X|Y = y]$, and by the same token, it also has a variance

$$var(X|Y = y) = E\left[(X - E[X|Y = y])^2 |Y = y\right].$$

Note that the conditional varience is itself a RV. (Why?)

Similar to the law of iterated expectation, one can develop a relation between $var(X)$ and $var(X|Y)$. This law is known as Law of Conditional Variances.

Law of Conditional Variances

$$\text{var}(X) = \mathbf{E}\big[\text{var}(X \mid Y)\big] + \text{var}\big(\mathbf{E}[X \mid Y]\big)$$

Proof:

To verify the law of conditional variances, we start with the identity

$$X - \mathbf{E}[X] = (X - \mathbf{E}[X \mid Y]) + (\mathbf{E}[X \mid Y] - \mathbf{E}[X])$$

We square both sides and then take expectations to obtain

$$\begin{aligned}\text{var}(X) &= \mathbf{E}\Big[(X - \mathbf{E}[X])^2\Big] \\ &= \mathbf{E}\Big[(X - \mathbf{E}[X \mid Y])^2\Big] + \mathbf{E}\Big[(\mathbf{E}[X \mid Y] - \mathbf{E}[X])^2\Big] \\ &\quad + 2\mathbf{E}\Big[(X - \mathbf{E}[X \mid Y])(\mathbf{E}[X \mid Y] - \mathbf{E}[X])\Big].\end{aligned}$$

Using the law of iterated expectations, the first term in the right-hand side of the above equation can be written as

$E\left[E\left[(X - E[X|Y])^2 |Y\right]\right] = E[var(X|Y)]$

The second term is $var(E[X|Y])$ since $E[X]$ is the mean of $E[X|Y]$.

So, we need to prove that the third term equals zero. Let $h(Y) = 2\,(E[X|Y] - E[X])$, the third term can be written as

$$\begin{aligned}\mathbf{E}\Big[(X - \mathbf{E}[X \mid Y])h(Y)\Big] &= \mathbf{E}\big[Xh(Y)\big] - \mathbf{E}\big[\mathbf{E}[X \mid Y]h(Y)\big] \\ &= \mathbf{E}\big[Xh(Y)\big] - \mathbf{E}\Big[\mathbf{E}\big[Xh(Y) \mid Y\big]\Big] \\ &= \mathbf{E}\big[Xh(Y)\big] - \mathbf{E}\big[Xh(Y)\big] \\ &= 0.\end{aligned}$$

4 Sum of a Random Number of RVs

- We consider here the RV $Y = (X_1 + X_2 + .. + X_N)$, where X_1, X_2, .., X_N are 'II' RV's with mean $\overline{X}$ and variance σ_X^2 and N is a discrete RV.
- We wish to **derive** formulas for the mean, variance, and the transform of Y. The derivation of these statistics is based on the concept of conditional distribution

$$\begin{aligned}\mathbf{E}[Y \mid N = n] &= \mathbf{E}[X_1 + \cdots + X_N \mid N = n]\\ &= \mathbf{E}[X_1 + \cdots + X_n \mid N = n]\\ &= \mathbf{E}[X_1 + \cdots + X_n]\\ &= n\mu.\end{aligned}$$

Hence, the conditional expectation can be expressed as

$$E_Y[Y|N] = N\overline{X}$$

By using the law of iterated expectation

$$E_Y[Y] = E_Y[E_N[Y|N]] = E_N[N\overline{X}] = \overline{X}E_N[N] = \overline{X}\ \overline{N}$$

Recall that $var(Y|N) = N\sigma_Y^2$ and using the law of conditional variance

$$\begin{aligned}\text{var}(Y) &= \mathbf{E}\big[\text{var}(Y \mid N)\big] + \text{var}\big(\mathbf{E}[Y \mid N]\big)\\ &= \mathbf{E}[N]\sigma^2 + \text{var}(N\mu)\\ &= \mathbf{E}[N]\sigma^2 + \mu^2\text{var}(N).\end{aligned}$$

The calculation of the transform proceeds along similar lines. The trans form associated with Y , conditional on N = n

$$\begin{aligned}\mathbf{E}[e^{sY} \mid N = n] &= \mathbf{E}\big[e^{sX_1} \cdots e^{sX_N} \mid N = n\big] = \mathbf{E}\big[e^{sX_1} \cdots e^{sX_n}\big]\\ &= \mathbf{E}[e^{sX_1}] \cdots \mathbf{E}[e^{sX_n}] = \big(M_X(s)\big)^n.\end{aligned}$$

Using the law of iterated expectations, the (unconditional) transform associated with Y is

$$\mathbf{E}[e^{sY}] = \mathbf{E}\big[\mathbf{E}[e^{sY} \mid N]\big] = \mathbf{E}\big[\big(M_X(s)\big)^N\big] = \sum_{n=0}^{\infty} (M_X(s))^n p_N(n).$$

This expression is similar to the transform $M_N(s)$ associated with N, except that z is replaced by $M_X(s)$.

EXAMPLE:
Consider estimating the mean, variance, and the transform of $Y = X_1 + X_2 + .. + X_N$, where N is a geometric RV with parameter p and X_i are IID exponential distribution with parameter λ
Solution
$E[Y] = E[X]E[N] = 1/(p\lambda)$

$$\begin{aligned}\text{var}(Y) &= \mathbf{E}[N]\text{var}(X) + (\mathbf{E}[X])^2\text{var}(N)\\ &= \frac{1}{p} \cdot \frac{1}{\lambda^2} + \frac{1}{\lambda^2} \cdot \frac{1-p}{p^2} = \frac{1}{\lambda^2 p^2}\end{aligned}$$

Note that $M_X(s) = \frac{\lambda}{\lambda+s}$ and $M_N(z) = \frac{z(1-p)}{1-(1-p)z}$
$M_Y(s) = \frac{pM_X(s)}{1-(1-p)M_X(s)} = \frac{p\lambda}{p\lambda+s}$

5 Homework

- Prove Theorem 1, Theorem 2, and Theorem 3
- Estimate the mean and variance of the packet delay from exiting the source until it is received at the destination in the Internet if the number of the links that the packet crosses follows a Geometric distribution with parameter p and the delay per link follows a hypo-exponential distribution with parameters μ_1 and μ_2.
- Derive the moment generating function for the number of arrivals from a poisson process (K) during a random interval X.