Modeling and Simulation
Lecture #2

Contents

1 Continuous Random Variables

1.1 Cumulative Distribution Function (CDF)

$$F_X(x) = P(X \leq x) \quad -\infty < x < \infty$$

- if $F_X(x)$ is a continuous function of x, then X is a continuous random variable.
- Properties of cumulative distribution function
  - $0 \leq F_X(x) \leq 1$, $-\infty < x < \infty$
  - $F_X(x)$ is a monotonically increasing func. of x
  - $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$
  - $P(X = c) = 0$ where c is any real number

1.2 Probability Density Function (PDF)

- $f(x) = \frac{dF_x(x)}{dx}$ is the pdf of X
- similarly, $F_X(x) = \int_{-\infty}^{x} f_X(x)dx$ for $-\infty < x < \infty$
- PDF properties
  - $f(x) \geq 0$ for all x
  - $\int_{-\infty}^{\infty} f_X(x)dx = 1$

1.3 Common Distributions

1.3.1 Uniform Distribution

- U(a,b) → pdf constant over the (a,b) interval and CDF is the ramp function

$$f(x) = \begin{cases} \frac{1}{b-a} & , a < x < b \\ 0 & , otherwise \end{cases}$$

$$F_X(x) = \begin{cases} 0 & , x < a \\ \frac{(x-a)}{(b-a)} & , a \leq x \leq b \\ 1 & , x > b \end{cases}$$

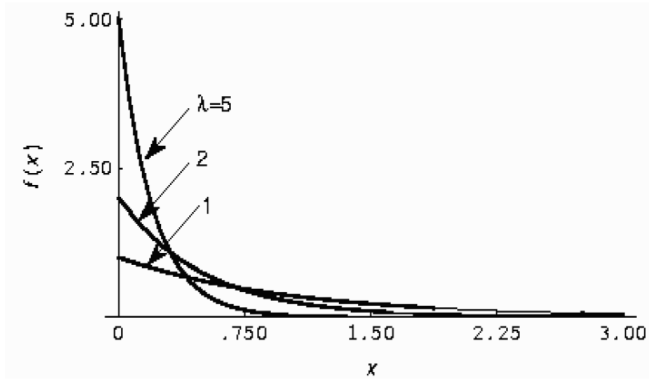
very common distribution for homogeneity

1.3.2 Exponential Distribution

- Arises commonly in reliability & queuing theory.
- A non-negative random variable
- It exhibits memoryless (Markov) property.
- Related to (the discrete) Poisson distribution
  - Interarrival time between two IP packets (or voice calls)
  - Time to failure, time to repair etc.

- Mathematically (CDF and pdf, respectively)

$$F_X(x) = 1 - e^{-\lambda x} \iff f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$$



- Exponential Memoryless property
  - Let $X > t$
  - Let $Y = X - t$, the remaining (residual) lifetime

$$\begin{aligned} P(Y \leq y | X > t) &= P(X \leq y + t | X > t) \\ &= \frac{P(X \leq y + t, X > t)}{P(X > t)} \\ &= 1 - e^{-\lambda y} \end{aligned}$$

  - The distribution of the remaining life, Y, does not depend on how long the component has been operating.
  - The minimum of two exponential distribution is another exponential distribution whose parameter is the sum of the parameters of the original distributions.
  - ***Racing property***: if two events with exponential distribution starts together, the probability that a specific event finishes first equals its parameter divided by the parameter sum.

1.3.3 Erlang-K distribution

- K ***identical*** exponential stages in ***series***

$$f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

$$F_{Y_k}(y) = 1 - \sum_{n=0}^{k-1} \frac{(\lambda y)^n e^{-\lambda y}}{n!}$$

- Very common in queuing systems with exponential inter-arrival and service time.

1.3.4 HypoExponential RV

- multiple Exp stages in ***series***.
- A general case for Erlang-K
- 2-stage Hypo-exp denoted as HYPO(μ_1, μ_2). The CDF and PDF are expressed as

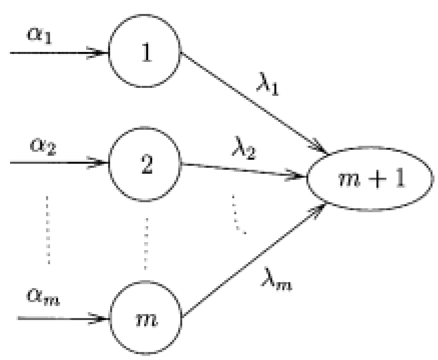
$$F_X(x) = 1 - \frac{\mu_2}{\mu_2 - \mu_1} e^{-\mu_1 x} + \frac{\mu_1}{\mu_2 - \mu_1} e^{-\mu_2 x}$$
$$f_X(x) = \frac{\mu_2 \mu_1}{\mu_2 - \mu_1} (e^{-\mu_1 x} - e^{-\mu_2 x})$$

- Disk service time may be modeled as a 3-stage Hypo-exponential as the overall time is the sum of the seek, the latency and the transfer time

1.3.5 Hyper-exponential Distribution

- Multiple Exp stages in parallel.

$$f_X(x) = \sum_{j=1}^{k} \alpha_j \lambda_j e^{-\lambda_j x}$$
$$F_X(x) = \sum_{j=1}^{k} \alpha_j (1 - e^{-\lambda_j x})$$



- A very good fit in modeling parameters with high variability. E.g., file length, residence time, ...

1.3.6 Gamma Distribution

$$f(t) = \frac{\lambda^\alpha t^{\alpha-1} e^{-\lambda t}}{\Gamma(\alpha)} \quad \alpha > 0, \ t > 0$$
$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

- Gamma with α = $\frac{1}{2}$ and = n/2 is known as the chi-square random variable with n degrees of freedom
- Represent a good distribution for modeling the mobility

1.3.7 Weibull Distribution

$$f(t) = \lambda \alpha t^{\alpha-1} e^{-\lambda t^{\alpha}}$$
$$F_T(t) = 1 - e^{-\lambda t^{\alpha}}$$

- α is called the shape parameter and λ is the scale parameter
- Frequently used to model fatigue failure, ball bearing failure etc. (very long tails)

1.3.8 Gaussian Distribution

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-0.5\left(\frac{x-\mu}{\sigma}\right)^2}$$

- μ: mean, σ: std. deviation, σ^2: variance (N(μ, σ2))
- Very common in statistical estimation/signal processing/communication theory etc.
- Central Limit Theorem: the sum of a large number of mutually independent RV's (having arbitrary distributions) starts following Normal distribution as n $\rightarrow \infty$
- Normal distribution N(0,1) is called normalized Gaussian

1.4 Distribution Fitting

- The three Exponential, hypo-exponential, and hyper-exponential distribution can be used to **approximate** any statistics
- In many applications, two phases for hypo and hyper-exponential distributions are sufficient to be a good approximation.
- Let C_x represents the coefficient of variation of a set of measurements/ samples.

$$C_x = \frac{\sigma_X}{\overline{X}}$$

where $\overline{X}$ is the average value of the measurements and σ_X is the standard deviation of the measurements

C_X	Best fit distribution	estimated parameters
=1	exponential	$\lambda = 1/\overline{X}$
<1	hypo-exponential	$\mu_1 = \frac{2}{\overline{X}}\left[1 + \sqrt{1 + 2(C_X^2 - 1)}\right]^{-1}$, $\mu_2 = \frac{2}{\overline{X}}\left[1 - \sqrt{1 + 2(C_X^2 - 1)}\right]^{-1}$
>1	hyper-exponential	choose α_1, $\alpha_2 = 1 - \alpha_1$ $\lambda_1 = \frac{1}{\overline{X}}\left[1 - \sqrt{\frac{\alpha_2}{\alpha_1}\frac{C_X^2-1}{2}}\right]^{-1}$ $\lambda_2 = \frac{1}{\overline{X}}\left[1 + \sqrt{\frac{\alpha_1}{\alpha_2}\frac{C_X^2-1}{2}}\right]^{-1}$

2 Expectations and Moments

2.1 Expectations

- There are several ways to abstract the information in the CDF into a single number: median, mode, mean

$$E(X) = \sum_i x_i p_i \quad for\, discrete\, RV$$

$$E(X) = \int x f_X(x) dx \quad for\, continuous\, RV$$

- The average is very common to use when showing performance metrics
- Example: the expected number of busy channels in a cellular system (system utilization)

2.2 Expectation Properties

- Scale and shift

$$E[aX + b] = aE[X] + b$$

- Linearity

$$E[X + Y] = E[X] + E[Y]$$

- Product of independent RVs

$$E[XY] = E[X]E[Y]$$

- Two RV are said to be uncorrelated iff $CoV[X, Y] = 0$ where

$$CoV[X, Y] = E[(X - E[X])`(Y - E[Y])]$$

  - NOTE: covariance measures the linear dependence.
    - Check: X˜U(-1,1) and Y=X2
      - Complete dependence
      - Uncorrelated

2.3 Moments

- consider a RV X and another RV $Y = \Delta(X)$ the

$$E[Y] = E[\Delta(X)] = \begin{cases} \sum_i \Delta(x_i) p_i & , discrete \\ \int \Delta(x) f_X(x) dx & continuous \end{cases}$$

- For $\Delta(X) = X^k$, $k = 1, 2, 3, ...,$ the $E[X^k]$ is called the $k^{th} moment$
- k=1 → mean , k=2 → second moment

- The second moments is used to calculate the variability of the RV (RV variance)

$$\begin{aligned} var(X) &= \sigma_X^2 = E[\left(X - \overline{X}\right)^2] \\ &= E[X^2] - \overline{X}^2 \end{aligned}$$

- The standard deviation is $\sigma_X = \sqrt{Var(X)}$
- Coefficient of variation $C_X = \sigma_X / \overline{X}$

3 Transform Methods

- Transforms are effective when dealing with compound random variables such as sums of independent RVs.
- The transforms can also be used to estimate different moments

$$M(s) = E[e^{-sX}] = \int_{-\infty}^{\infty} e^{-sx} f_x(x) dx, \quad , continuous$$

- Transforms and moments

$$E[X^k] = \begin{cases} (-1)^k \frac{d^k M}{ds^k}|_{s=0} & continuous \\ \frac{d^k M}{dz^k}|z = 1 & discrete \end{cases}$$

Remember if two RVs have the same M(s) then they have the same distribution

The transform of SUM of mutually independent RVs is the product of the RV transforms (Convolution-product property)

4 Function of a Random Variable

4.1 Function of one RV

- typically appear on processing a RV in your analysis or design
- The function of any RV is considered a new RV (why?)
- the objective here is to determine the distribution of the new RV
- Let $Y = \Phi(X)$ such that $\Phi(X)$ is monotone and differentiable

$$\begin{aligned} F_Y(y) &= Pr(Y \leq y) \\ &= Pr(\Phi(X) < y) \\ &= Pr(X \leq \Phi^{-1}(y)) \\ &= F_X(\Phi^{-1}(y)) \end{aligned}$$

where $\Phi^{-1}(.)$ is the inverted function of $\Phi(.)$. Hence, one can write

$$f_Y(y) = f_X(\Phi^{-1}(y)) \frac{d}{dy} \Phi^{-1}(y)$$

- **Example:** $Y = \frac{-ln(1-X)}{\lambda}$ and X is U(0,1)
X=1-$e^{-\lambda y}$→ $f_Y(y) = \lambda e^{-\lambda y}$, $y \in [0, \infty]$

5 Homework

- Prove that the minimum of two exponential distribution is another exponential distribution whose parameter is the sum of the parameters of the original distributions.
- Prove the racing probability of exponential distribution
- Develop a model for the file size knowing that the average file size is 5Kbytes and the standard deviation of the file size is 10Kbytes.
- Develop a model for the packet delay in a local area network given that the average packet delay is 1.2sec and its standard deviation is 0.7sec.
- Derive the first and second moments for the studied continuous and discrete distributions
- Prove the expectation properties

6 References

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- Kishor Trivedi, "Probability and Statistics with Reliability, Queuing, and Computer Systems Applications"
- B. R. Havercort, "Performance of Computer Communication Systems, a model based approach"